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Process-independent determination of two-loop electroweak next-to-leading logarithms

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Nucl. Phys. B761 (2007) 1–62 [arXiv:hep-ph/0608326] (massless fermions)
JHEP 11 (2008) 062 [arXiv:0809.0800 [hep-ph]] (heavy quarks)

Overview

I Electroweak (EW) corrections at high energies

- origin and importance of EW logs
- existing virtual EW 2-loop corrections

II Two-loop next-to-leading logarithmic (NLL) corrections

- extraction of NLL contributions
- evaluation of Feynman diagrams:
  expansion by regions & Mellin–Barnes representation

III Results for processes involving massless and massive fermions

- factorization and exponentiation of the logs
- comparisons and applications
- structure of the result to all orders in ϵ

IV Summary & outlook

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I Electroweak corrections at high energies

Precision collider physics

Precision measurements at colliders and theoretical predictions with electroweak (EW) + QCD corrections enable us to test the Standard Model at various energy scales:

- up to now (LEP, Tevatron) at energy scales $\lesssim M_{\mathrm{W,Z}}$
- upcoming colliders (LHC, ILC/CLIC) → reach TeV regime
  ↪ new energy domain $\sqrt{s} \gg M_{\mathrm{W}}$ becomes accessible!

EW radiative corrections at high energies $\sqrt{s} \gg M_{\mathrm{W}}$

⇒ enhanced by large Sudakov logarithms

$$\text{per loop:} \quad \ln^2\left(\frac{s}{M_{\mathrm{W}}^2}\right) \sim 25 \quad \text{at } \sqrt{s} \sim 1\,\text{TeV}$$

↪ corrections rise with energy

Origin of large EW logs

- mass singularities: real or virtual emission of soft/collinear gauge bosons from external particles
- remnants from UV singularities

Massless gauge bosons

real emission of soft/collinear photons/gluons cannot be detected separately

$\hookrightarrow$ mass singularities cancel between real & virtual corrections (KLN theorem)

Massive gauge bosons

real emission of W's, Z's can (in principle) be detected separately

$\hookrightarrow$ only virtual corrections: large logs remain present in exclusive observables,

$\hookrightarrow$ even in inclusive observables (Bloch–Nordsieck violations)

General form of virtual EW corrections for $s \gg M_\mathrm{W}^2$

$$\left[L = \ln\left(\frac{s}{M_\mathrm{W}^2}\right)\right]$$

$\hookrightarrow$ LL (leading logarithmic), NLL (next-to-leading logarithmic), $\mathrm{N^2LL}$ … terms:

1 loop:
$$\alpha\left[\underset{\displaystyle -17\,\%}{\underset{\downarrow}{C_1^{\mathrm{LL}}\,L^2}} + \underset{\displaystyle +12\,\%}{\underset{\downarrow}{C_1^{\mathrm{NLL}}\,L}} + \underset{\displaystyle -3\,\%}{\underset{\downarrow}{C_1^{\mathrm{N^2LL}}}}\right] + \mathcal{O}\left(\frac{M_\mathrm{W}^2}{s}\right)$$

2 loops:
$$\alpha^2\left[\underset{\displaystyle +1.7\,\%}{\underset{\downarrow}{C_2^{\mathrm{LL}}\,L^4}} + \underset{\displaystyle -1.8\,\%}{\underset{\downarrow}{C_2^{\mathrm{NLL}}\,L^3}} + \underset{\displaystyle +1.2\,\%}{\underset{\downarrow}{C_2^{\mathrm{N^2LL}}\,L^2}} + \underset{\displaystyle -0.3\,\%}{\underset{\downarrow}{C_2^{\mathrm{N^3LL}}\,L}} + C_2^{\mathrm{N^4LL}}\right] + \mathcal{O}\left(\frac{M_\mathrm{W}^2}{s}\right)$$

[$\sigma(u\bar{u} \to d\bar{d})$ @ $\sqrt{s} = 1$ TeV, B.J., Kühn, Penin, Smirnov ’05]

For theoretical predictions with accuracy $\sim 1\,\%$:

$\Rightarrow$ 2-loop corrections important

$\Rightarrow$ LL approximation not sufficient

With massless photons: log $\rightsquigarrow$ $1/\epsilon$ in $D = 4 - 2\epsilon$ dimensions

Existing virtual EW 2-loop corrections

Resummation of 1-loop results using evolution equations valid for unbroken SU(2)×U(1) ($M_\gamma = M_{\mathrm{Z}} = M_{\mathrm{W}}$) and for pure QED:

$$\alpha^2 \left[\underbrace{\underbrace{C_{\mathsf{LL}} \ln^4\left(\frac{s}{M_{\mathrm{W}}^2}\right)}_{\text{Fadin, Lipatov, Martin, Melles '99}} + \underbrace{C_{\mathsf{NLL}} \ln^3\left(\frac{s}{M_{\mathrm{W}}^2}\right)}_{\text{Melles '00, '01}}}_{\text{arbitrary processes}} + \underbrace{\underbrace{C_{\mathsf{N^2LL}} \ln^2\left(\frac{s}{M_{\mathrm{W}}^2}\right)}_{\text{Kühn, Moch, Penin, Smirnov '99–'01}} + \underbrace{C_{\mathsf{N^3LL}} \ln\left(\frac{s}{M_{\mathrm{W}}^2}\right)}_{\text{B.J., Kühn, Moch, Penin, Smirnov '03–'05}}}_{\text{massless } f\,\bar{f} \to f'\,\bar{f}' \text{ processes}} \right]$$

+ SCET approach [Chiu, Golf, Kelley, Manohar '07, '08]
+ N^{2}LL for $e^+ e^- \to W^+ W^-$ [Kühn, Metzler, Penin '07]

Explicit 2-loop calculations based on spontaneously broken SU(2)×U(1), $M_{\mathrm{Z}} \neq M_{\mathrm{W}}$:

$$\alpha^2 \left[\underbrace{\underbrace{C_{\mathsf{LL}} \ln^4\left(\frac{s}{M_{\mathrm{W}}^2}\right)}_{\text{Melles '00; Hori, Kawamura, Kodaira '00; Beenakker, Werthenbach '00, '01}} + \underbrace{C_{\mathsf{NLL}}^{\mathsf{ang}} \ln\left(\frac{t}{s}\right) \ln^3\left(\frac{s}{M_{\mathrm{W}}^2}\right)}_{\text{Denner, Melles, Pozzorini '03}}}_{\text{arbitrary processes}} + \underbrace{\underbrace{C_{\mathsf{NLL}}^{\mathsf{rem}} \ln^3\left(\frac{s}{M_{\mathrm{W}}^2}\right)}_{\text{Pozzorini '04; Denner, B.J., Pozzorini '06, '08}}}_{n\text{-fermion processes}} \right]$$

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II Two-loop next-to-leading logarithmic corrections

Goal: virtual 2-loop EW corrections for arbitrary processes in NLL accuracy

Process-independent: separate loop integrals from Born matrix elements

↪ already completed: processes involving massless & massive external fermions

Parameters: $[D = 4 - 2\epsilon]$

- different large kinematical invariants $r_{ij} = (p_i + p_j)^2 \sim Q^2 \gg M_\mathrm{W}^2$
- different heavy particle masses $M_\mathrm{W}^2 \sim M_\mathrm{Z}^2 \sim m_\mathrm{t}^2 \sim M_\mathsf{Higgs}^2$
- massive top quark, other fermions massless

⇒ logs $L = \ln\left(\frac{Q^2}{M_\mathrm{W}^2}\right)$ and $\frac{1}{\epsilon}$ poles (from virtual photons, counted like logs)

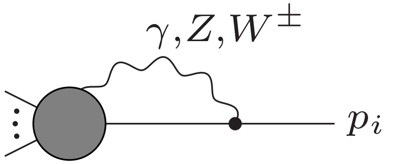
1 loop: LL $\to \epsilon^{-2},\ L\epsilon^{-1},\ L^2,\ L^3\epsilon,\ L^4\epsilon^2;$ NLL $\to \epsilon^{-1},\ L,\ L^2\epsilon,\ L^3\epsilon^2$

2 loops: LL $\to \epsilon^{-4},\ L\epsilon^{-3},\ L^2\epsilon^{-2},\ L^3\epsilon^{-1},\ L^4;$ NLL $\to \epsilon^{-3},\ L\epsilon^{-2},\ L^2\epsilon^{-1},\ L^3$

⇒ NLL coefficients involve small logs $\ln\left(\frac{-r_{ij}}{Q^2}\right)$ and $\ln\left(\frac{M_\mathrm{Z}^2, m_\mathrm{t}^2}{M_\mathrm{W}^2}\right)$

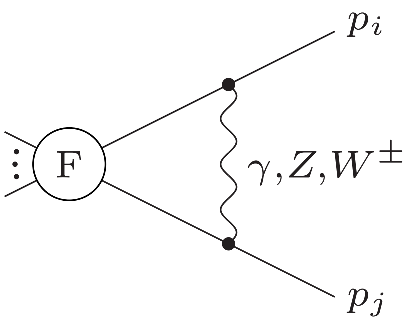
Extraction of NLL contributions

Logs originate from mass singularities when a virtual gauge boson ($\gamma, Z, W^\pm$) couples to an on-shell external leg
→ single log from collinear region (+ UV logs)



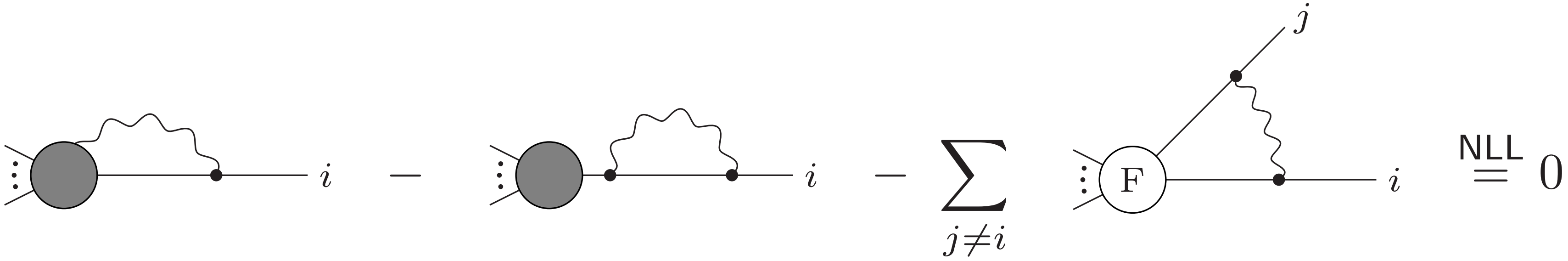
Isolate factorizable contributions:
gauge boson exchanged between external legs;

separate loop integral from Born diagram Ⓕ
via soft–collinear approximation
→ double log from soft & collinear region



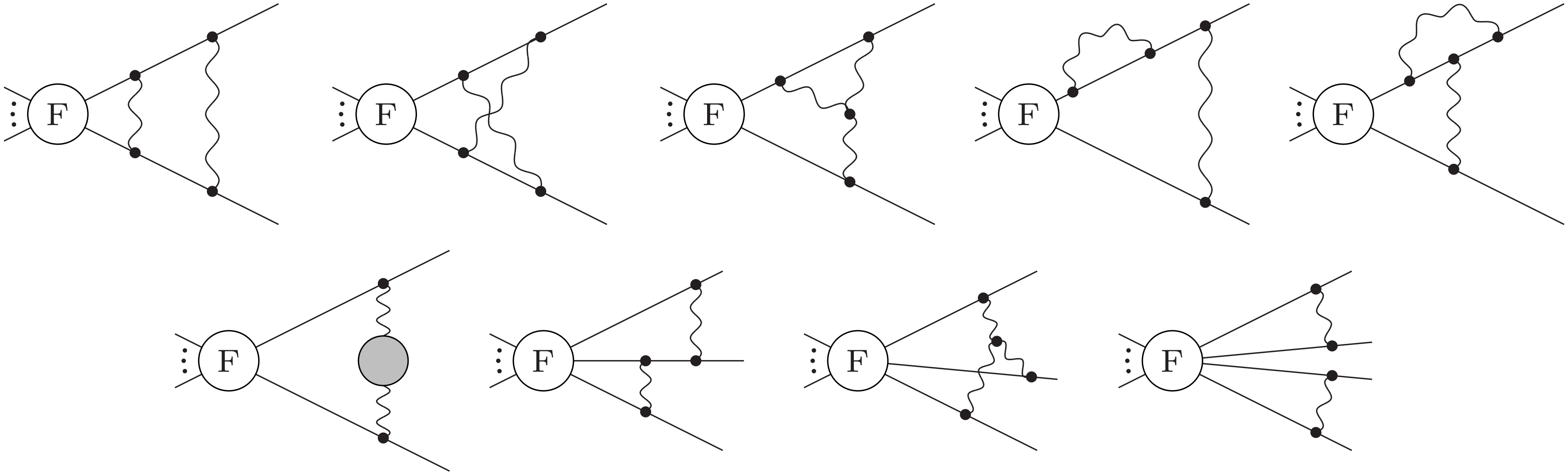
Remaining non-factorizable contributions: collinear Ward identities

Denner, Pozzorini '00, '01



The factorizable contributions contain all soft or collinear NLL mass singularities.

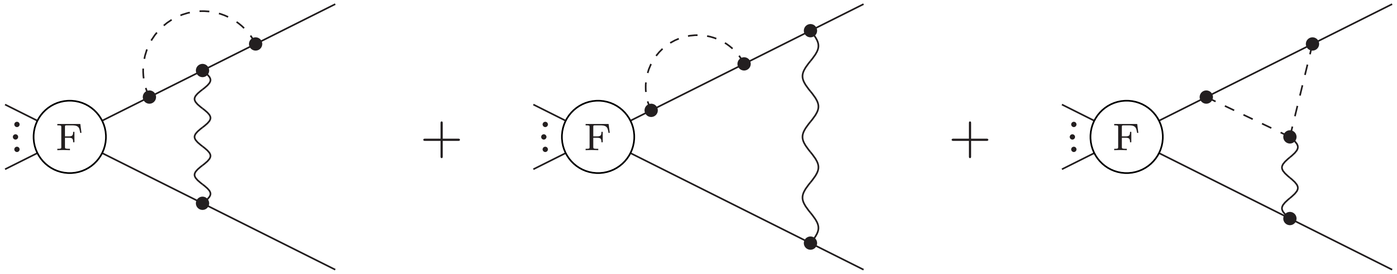
Factorizable contributions at 2 loops:



↪ non-factorizable contributions vanish

Yukawa couplings of massive fermions to Higgs & Goldstone bosons

↪ only three non-suppressed factorizable diagrams:



$\overset{\text{NLL}}{=} 0$

⇒ Sum vanishes due to gauge invariance of Yukawa interaction

⇝ Yukawa interaction contributes only to wave-function renormalization

Evaluation of Feynman diagrams

Need to evaluate Feynman diagrams in the high-energy limit $Q^2 \gg M^2 \sim M_{\mathrm{W}}^2$
⇒ discuss combination of two methods:

- expansion by regions,
- Mellin–Barnes representations.

cf. B.J., Smirnov '06 & refs. therein

The same diagrams have also been evaluated independently with an automatized algorithm based on sector decomposition (not discussed here). Denner, Pozzorini '04

Expansion by regions

Beneke, Smirnov '98

- **problem:** expansion of a Feynman integral in a limit like $Q^2 \gg M^2$
- **wanted:** expansion of the integrand before integration
- **complication:** expansion and integration do <u>not</u> commute, expansion creates new singularities

Recipe for the method of expansion by regions:

1. within the integration domain for the loop momenta, consider the relevant *regions* (usually around points where singularities arise)
2. in every region, *expand* the integrand in a *Taylor series* with respect to the parameters that are considered small there (logarithmic approximation: just set small parameters to zero)
3. *integrate* each of the expanded integrands over the *whole integration domain*
4. set to zero any *integral without scale* (like with dimensional regularization)

For some individual regions, in addition to ϵ, analytic regularization is needed (drops out in sum of regions).

Expansion by regions with massive external legs

Expansion before integration:

1. expansion eliminates small terms with respect to large terms
2. integration produces new small, but possibly finite invariants
   $p_i^2, p_j^2 \in \{0, m_\mathrm{t}^2\} \ll 2p_i \cdot p_j \sim Q^2$

Need to make these invariants explicit before integration

$\Rightarrow$ shift to lightlike momenta: $p_{i,j} = \tilde{p}_{i,j} + \dfrac{p_{i,j}^2}{2\tilde{p}_i \cdot \tilde{p}_j}\, \tilde{p}_{j,i}$ with $\tilde{p}_i^2 = \tilde{p}_j^2 = 0$

Relevant regions for each loop momentum k: $[M \sim M_\mathrm{W,Z} \sim m_\mathrm{t} \ll Q]$

	$k_{\parallel \tilde{p}_i}$	$k_{\parallel \tilde{p}_j}$	$k_{\perp(\tilde{p}_i,\tilde{p}_j)}$
hard	Q	Q	Q
soft	M	M	M
ultrasoft	M^2/Q	M^2/Q	M^2/Q

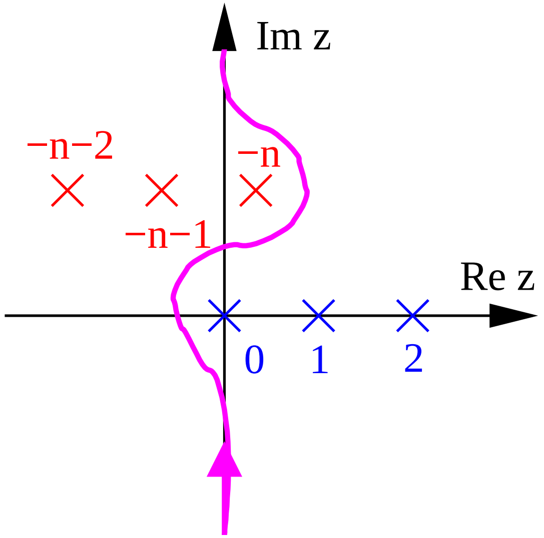
	$k_{\parallel \tilde{p}_i}$	$k_{\parallel \tilde{p}_j}$	$k_{\perp(\tilde{p}_i,\tilde{p}_j)}$	
i-collinear	Q	M^2/Q	M	
j-collinear	M^2/Q	Q	M	
i-ultracollinear	M^2/Q	M^4/Q^3	M^3/Q^2	new!
j-ultracollinear	M^4/Q^3	M^2/Q	M^3/Q^2	new!

Mellin–Barnes representation

Ussyukina '75; Boos, Davydychev '91

Feynman integrals with many parameters are hard to evaluate
$\hookrightarrow$ separate parameters by Mellin–Barnes (MB) representation:

$$\frac{1}{(A+B)^n} = \frac{1}{\Gamma(n)} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i}\, \Gamma(n+z)\, \Gamma(-z)\, \frac{B^z}{A^{n+z}}$$



- MB integrals go along the imaginary axis, leaving poles of $\Gamma(z+\ldots)$ to the left and poles of $\Gamma(-z+\ldots)$ to the right of the integration contour
- evaluation: close the integration contour to the right ($|B| \le |A|$) or to the left ($|B| \ge |A|$) and pick up the residues within the contour: $\operatorname{Res}\Gamma(z)\big|_{z=-i} = (-1)^i/i!$
  $\hookrightarrow$ asymptotic expansion in powers of (B/A) or (A/B) and $\ln(A/B)$
- closely related to *expansion by regions*: contributions from residues of MB integral correspond to contributions from regions

This project:

- use MB representation for expressions originating from expansion by regions
- extract and evaluate singularities from MB integrals

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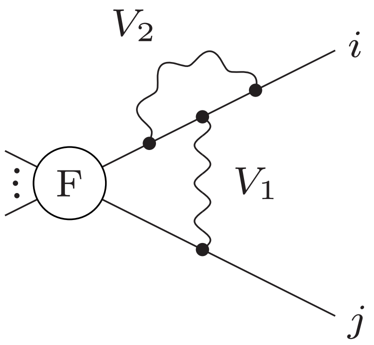
Example for the contribution of a Feynman diagram



$$\overset{\text{NLL}}{=} \underbrace{\sum_{V_1,V_2=\gamma,Z,W^\pm}}_{\text{sum over gauge bosons}} \overbrace{D(M_{V_1},M_{V_2};p_i,p_j)}^{\text{scalar 2-loop integral}} \underbrace{I_i^{V_2} I_i^{V_1} I_i^{\bar{V}_2} I_j^{\bar{V}_1}}_{\substack{\text{isospin matrices}\\ \text{applied to external legs}}} \quad \overset{\substack{\text{Born amplitude}\\ \text{factorized}\\ \uparrow}}{\mathcal{M}_0}$$

All relevant combinations of $\left\{\begin{matrix}\text{massless}\\ \text{massive}\end{matrix}\right\}$ $\left\{\begin{matrix}\text{external}\\ \text{internal}\end{matrix}\right\}$ fermions evaluated explicitly!

Result for amplitude of fermionic processes $f_1 f_2 \to f_3 \cdots f_n$ in $\mathcal{O}(\alpha^2)$

$$\mathcal{M} \overset{\text{NLL}}{=} \underbrace{\exp\big(\Delta F^{\text{em}}\big)}_{\substack{\text{electromagnetic}\\ M_\gamma = 0}} \times \underbrace{\exp\big(F^{\text{sew}}\big)}_{\substack{\text{symmetric-electroweak}\\ M_\gamma = M_\text{Z} = M_\text{W}}} \times \underbrace{\big(1+\Delta F^{\text{Z}}\big)}_{\substack{\text{corrections}\\ \text{from } M_\text{Z} \neq M_\text{W}}} \times \underbrace{\mathcal{M}_0}_{\text{Born}}$$

- universal result: F^{sew}, ΔF^{em}, ΔF^{Z} depend only on external quantum numbers
- electromagnetic singularities (in ΔF^{em}) factorized → separable

$$\mathcal{M} \overset{\text{NLL}}{=} \exp\big(\Delta F^{\text{em}}\big) \;\times\; \exp\big(F^{\text{sew}}\big) \;\times\; \big(1 + \Delta F^{\text{Z}}\big) \;\times\; \mathcal{M}_0$$

Symmetric-electroweak terms: independent of fermion masses

$$F^{\text{sew}} = \frac{1}{2}\sum_{i=1}^{n}\Bigg\{-\frac{\alpha}{4\pi}\Bigg[\sum_{j\neq i}\sum_{V=\gamma,Z,W^\pm} I_i^{\bar V} I_j^V\, I_{ij}(\epsilon, M_{\text{W}}) + \overbrace{\frac{z_i^{\text{Yuk}}\, m_{\text{t}}^2}{4 s_{\text{w}}^2 M_{\text{W}}^2}\bigg(L + \frac{1}{2}L^2\epsilon + \frac{1}{6}L^3\epsilon^2 + \mathcal{O}(\epsilon^3)\bigg)}^{\text{Yukawa contribution}}\Bigg]$$

$$+ \left(\frac{\alpha}{4\pi}\right)^2 \left[\frac{b_1^{(1)}}{c_{\text{w}}^2}\left(\frac{Y_i}{2}\right)^2 + \frac{b_2^{(1)}}{s_{\text{w}}^2} C_i^{\text{w}}\right] J_{ii}(\epsilon, M_{\text{W}}, \mu_{\text{R}}^2)\Bigg\},$$

$$I_{ij}(\epsilon, M_{\text{W}}) \overset{\text{NLL}}{=} -L^2 - \frac{2}{3}L^3\epsilon - \frac{1}{4}L^4\epsilon^2 + \left[3 - 2\ln\left(\frac{-r_{ij}}{Q^2}\right)\right]\left(L + \frac{1}{2}L^2\epsilon + \frac{1}{6}L^3\epsilon^2\right) + \mathcal{O}(\epsilon^3),$$

$$J_{ij}(\epsilon, M_{\text{W}}, \mu^2) = \frac{1}{\epsilon}\left[I_{ij}(2\epsilon, M_{\text{W}}) - \left(\frac{Q^2}{\mu^2}\right)^{\epsilon} I_{ij}(\epsilon, M_{\text{W}})\right]$$

Terms from $M_{\text{Z}} \neq M_{\text{W}}$:

$$\Delta F^{\text{Z}} \overset{\text{NLL}}{=} \frac{1}{2}\sum_{i=1}^{n}\frac{\alpha}{4\pi}\,(I_i^{\text{Z}})^2\, \overbrace{\ln\left(\frac{M_{\text{Z}}^2}{M_{\text{W}}^2}\right)\big(2L + 2L^2\epsilon + L^3\epsilon^2\big)}^{=I_{ii}(\epsilon,M_{\text{Z}})-I_{ii}(\epsilon,M_{\text{W}})} + \mathcal{O}(\epsilon^3)$$

$$\mathcal{M} \overset{\text{NLL}}{=} \exp\bigl(\Delta F^{\text{em}}\bigr) \times \exp\bigl(F^{\text{sew}}\bigr) \times \bigl(1 + \Delta F^{\mathrm{Z}}\bigr) \times \mathcal{M}_0$$

Electromagnetic terms:

$[\mu_{\mathrm{R}}^2 = M_{\mathrm{W}}^2]$

↪ correspond to (QED with $M_\gamma = 0$) minus (QED with $M_\gamma = M_{\mathrm{W}}$)

$$\Delta F^{\text{em}} = \frac{1}{2} \sum_{i=1}^{n} \Biggl\{ -\frac{\alpha}{4\pi} \sum_{j \neq i} Q_i Q_j \Bigl[I_{ij}(\epsilon, 0) - I_{ij}(\epsilon, M_{\mathrm{W}}) \Bigr] + \left(\frac{\alpha}{4\pi}\right)^2 b_{\text{QED}}^{(1)} Q_i^2 \Bigl[J_{ii}(\epsilon, 0, M_{\mathrm{W}}^2) - J_{ii}(\epsilon, M_{\mathrm{W}}, M_{\mathrm{W}}^2) \Bigr] \Biggr\},$$

$$I_{ij}(\epsilon, 0) \overset{\text{NLL}}{=} -\left[3 - 2\ln\left(\frac{-r_{ij}}{Q^2}\right) \right] \epsilon^{-1} + \Biggl\{ \overbrace{-\delta_{i,0}\, \epsilon^{-2} + \delta_{i,\mathrm{t}}}^{\text{dependence on fermion mass } m_i} \Biggl[L\epsilon^{-1} + \frac{1}{2} L^2 + \frac{1}{6} L^3 \epsilon + \frac{1}{24} L^4 \epsilon^2 + \left(\frac{1}{2} - \ln\left(\frac{m_i^2}{M_{\mathrm{W}}^2}\right) \right) \left(\epsilon^{-1} + L + \frac{1}{2} L^2 \epsilon + \frac{1}{6} L^3 \epsilon^2 \right) \Biggr] + (i \to j) \Biggr\} + \mathcal{O}(\epsilon^3),$$

$$J_{ij}(\epsilon, 0, \mu^2) = \frac{1}{\epsilon} \left[I_{ij}(2\epsilon, 0) - \left(\frac{Q^2}{\mu^2}\right)^{\epsilon} I_{ij}(\epsilon, 0) \right]$$

Comparisons and applications

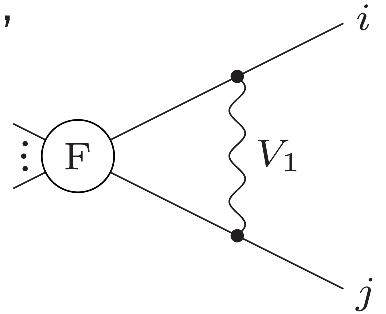
Comparison to existing results

- previous results for form factor and angular-dependent NLLs reproduced and extended Denner, Melles, Pozzorini '03; Pozzorini '04
- agreement with general resummation predictions based on evolution equations Melles '00, '01
- agreement with SCET results Chiu, Golf, Kelley, Manohar '07, '08

Application to 4-fermion scattering

- neutral current $f\bar{f} \to f'\bar{f}'$: NLL-agreement with massless-fermion N^{3}LL calculation, additional fermion-mass effects B.J., Kühn, Penin, Smirnov '05
- charged current $f_1\bar{f}_2 \to f_3\bar{f}_4$: new NLL result
- also applicable to processes with fermions and gluons, e.g. $g\,g \to f\,\bar{f}$: gluons $=$ legs with zero EW quantum numbers

Structure of the result to all orders in ϵ



Expansion by regions: look at contributions from individual regions, e.g. in 1-loop diagram: [with subtraction of UV $1/\epsilon$ pole]

hard region: $-2\left(\epsilon^{-2}+2\epsilon^{-1}\right)\left(\frac{Q^2}{-r_{ij}}\right)^{\epsilon}$

collinear regions:

- $m_1 = 0 \Rightarrow \left(\epsilon^{-2}+2\epsilon^{-1}\right)\left[\left(\frac{Q^2}{m_i^2}\right)^{\epsilon}+\left(\frac{Q^2}{m_j^2}\right)^{\epsilon}\right]$
- $m_1 \neq 0 \Rightarrow 2\left[\epsilon^{-2}-\underbrace{\ln\left(\frac{-r_{ij}}{m_1^2}\right)}_{\text{finite remainder from singularity cancelled between i-/j-collinear regions}}\epsilon^{-1}+2\epsilon^{-1}\right]\left(\frac{Q^2}{m_1^2}\right)^{\epsilon}$

Each region depends on mass parameters via **one unique power** of $(Q^2/m^2)^{\epsilon}$.

↪ Logs $\ln(Q^2/m^2)$ are generated by poles ϵ^{-n} in prefactor.

↪ Additional logarithms arise from singularities cancelled between collinear regions.

↪ $\mathcal{O}(\epsilon^0)$ in prefactor is beyond NLL accuracy.

⇝ In NLL accuracy this representation is valid **to all orders in ϵ**!

NLL result to all orders in ϵ

$$\mathcal{M} \overset{\text{NLL}}{=} \exp\big(\Delta F^{\text{em}}\big) \times \exp\big(F^{\text{sew}}\big) \times \big(1 + \Delta F^{\text{Z}}\big) \times \mathcal{M}_0$$

Every part of the result is known to all orders in ϵ. Most important ingredient:

$$I_{ij}(\epsilon, m_1) \overset{\text{NLL}}{=} -\left(2\epsilon^{-2} + 3\epsilon^{-1}\right) z_{ij}^{-\epsilon} + \left\{ \left[2\epsilon^{-2} - 2L\epsilon^{-1} + (3 - 2l_{ij} + 2l_1)\,\epsilon^{-1}\right] z_1^{-\epsilon} + \delta_{1,\gamma} \left(\epsilon^{-2} + \frac{1}{2}\epsilon^{-1}\right) \left(z_i^{-\epsilon} + z_j^{-\epsilon}\right) \right\} Z^{\epsilon}$$

with

$$Z = \frac{Q^2}{M_{\text{W}}^2},\ L = \ln Z;\ z_{ij} = \frac{-r_{ij}}{Q^2},\ l_{ij} = \ln z_{ij};\ z_a = \frac{m_a^2}{M_{\text{W}}^2},\ l_a = \ln z_a,\ a = 1, 2, \ldots, i, j, \ldots$$

$z_a^{n\epsilon} \equiv 0$ if $m_a = 0,\ n \neq 0$; $\qquad \delta_{a,\gamma} = \begin{cases} 1, & m_a = 0 \\ 0, & m_a \sim M_{\text{W}} \end{cases}$

Exponentiation

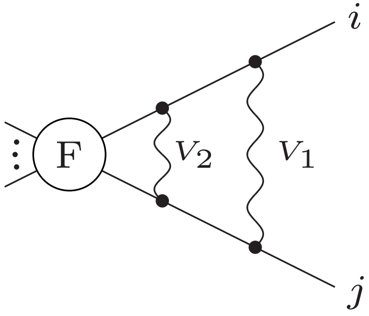
1-loop ⇝ $I_{ij}(\epsilon, m_1)$: Z^0 (hard), Z^{ϵ} (collinear)

2-loop ⇝ $I_{ij}(\epsilon, m_1) \times I_{kl}(\epsilon, m_2)$, $I_{ij}(2\epsilon, m_1)$, $Z^{\epsilon} \times I_{ij}(\epsilon, m_1)$:
Z^0 (hard–hard), Z^{ϵ} (hard–collinear), $Z^{2\epsilon}$ (collinear–collinear)

2-loop contributions to all orders in ϵ

2-loop result involves Z^0 (hard–hard), Z^ϵ (hard–collinear), $Z^{2\epsilon}$ (collinear–collinear)

But:

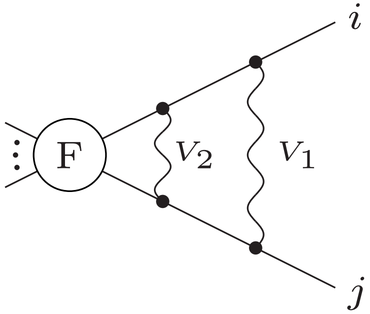


$$
\begin{aligned}
&\hookrightarrow \left(\epsilon^{-4}+4\epsilon^{-3}\right) z_{ij}^{-2\epsilon}\\
&+\left\{4\left[\epsilon^{-4}+L\epsilon^{-3}+\left(l_{ij}-l_1\right)\epsilon^{-3}+2L\epsilon^{-2}\right] z_1^{-\epsilon}-\frac{2}{3}\delta_{1,\gamma}\left(\epsilon^{-4}+4\epsilon^{-3}\right)\left(z_i^{-\epsilon}+z_j^{-\epsilon}\right)\right\} z_{ij}^{-\epsilon} Z^{\epsilon}\\
&+\left\{-\left[5\epsilon^{-4}-2L\epsilon^{-3}+2\left(2-l_{ij}+l_1\right)\epsilon^{-3}\right] z_1^{-2\epsilon}\right.\\
&\qquad\left.+\,\delta_{1,\gamma}\left[-\left(\epsilon^{-4}-2L\epsilon^{-3}+2\left(2-l_{ij}+l_2\right)\epsilon^{-3}\right) z_2^{-2\epsilon}+\left(\epsilon^{-4}+2\epsilon^{-3}\right) z_2^{-\epsilon}\left(z_i^{-\epsilon}+z_j^{-\epsilon}\right)\right]\right\} Z^{2\epsilon}\\
&+\delta_{1,\gamma}\left\{-\frac{1}{3}\left(\epsilon^{-4}-2\epsilon^{-3}\right) z_2^{-3\epsilon}\left(z_i^{-\epsilon}+z_j^{-\epsilon}\right)+\frac{1}{6}\delta_{2,\gamma}\left(\epsilon^{-4}+4\epsilon^{-3}\right)\left(z_i^{-\epsilon}z_j^{-3\epsilon}+z_j^{-\epsilon}z_i^{-3\epsilon}\right)\right\} z_{ij}^{2\epsilon} Z^{4\epsilon}
\end{aligned}
$$

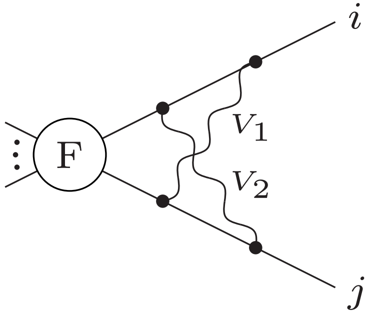
new contribution: $Z^{4\epsilon}$ (ultracollinear–collinear), not present in 2-loop result?!

$\hookrightarrow$ need non-trivial cancellation of all $Z^{4\epsilon}$ terms in total amplitude!

Cancellation of $Z^{4\epsilon}$ terms in 2-loop result

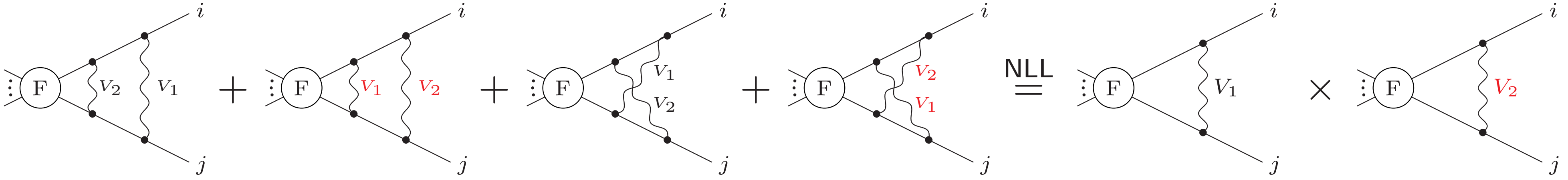


$$\rightsquigarrow \quad \delta_{1,\gamma}\left\{-\frac{1}{3}\left(\epsilon^{-4}-2\epsilon^{-3}\right)z_2^{-3\epsilon}\left(z_i^{-\epsilon}+z_j^{-\epsilon}\right) + \frac{1}{6}\delta_{2,\gamma}\left(\epsilon^{-4}+4\epsilon^{-3}\right)\left(z_i^{-\epsilon}z_j^{-3\epsilon}+z_j^{-\epsilon}z_i^{-3\epsilon}\right)\right\} z_{ij}^{2\epsilon}\, Z^{4\epsilon}$$



$$\rightsquigarrow \quad \left\{\frac{1}{3}\left(\epsilon^{-4}-2\epsilon^{-3}\right)\left(\delta_{1,\gamma}z_2^{-3\epsilon}z_i^{-\epsilon}+\delta_{2,\gamma}z_1^{-3\epsilon}z_j^{-\epsilon}\right) - \frac{1}{6}\delta_{1,\gamma}\delta_{2,\gamma}\left(\epsilon^{-4}+4\epsilon^{-3}\right)\left(z_i^{-\epsilon}z_j^{-3\epsilon}+z_j^{-\epsilon}z_i^{-3\epsilon}\right)\right\} z_{ij}^{2\epsilon}\, Z^{4\epsilon}$$

$\Rightarrow$ $Z^{4\epsilon}$ terms cancel in combination of scalar loop integrals:



$\hookrightarrow$ this relation (and others) checked to all orders in ϵ ✓

Overview

I Electroweak (EW) corrections at high energies

- origin and importance of EW logs
- existing virtual EW 2-loop corrections

II Two-loop next-to-leading logarithmic (NLL) corrections

- extraction of NLL contributions
- evaluation of Feynman diagrams:
  expansion by regions & Mellin–Barnes representation

III Results for processes involving massless and massive fermions

- factorization and exponentiation of the logs
- comparisons and applications
- structure of the result to all orders in ϵ

IV Summary & outlook

IV Summary & outlook

Massless and massive fermionic processes $f_1 f_2 \to f_3 \cdots f_n$ (+ gluons) with $(p_i + p_j)^2 \gg M_\mathrm{W}^2$ and different masses $M_\mathrm{W}^2 \sim M_\mathrm{Z}^2 \sim m_\mathrm{t}^2 \sim M_\mathrm{Higgs}^2$:

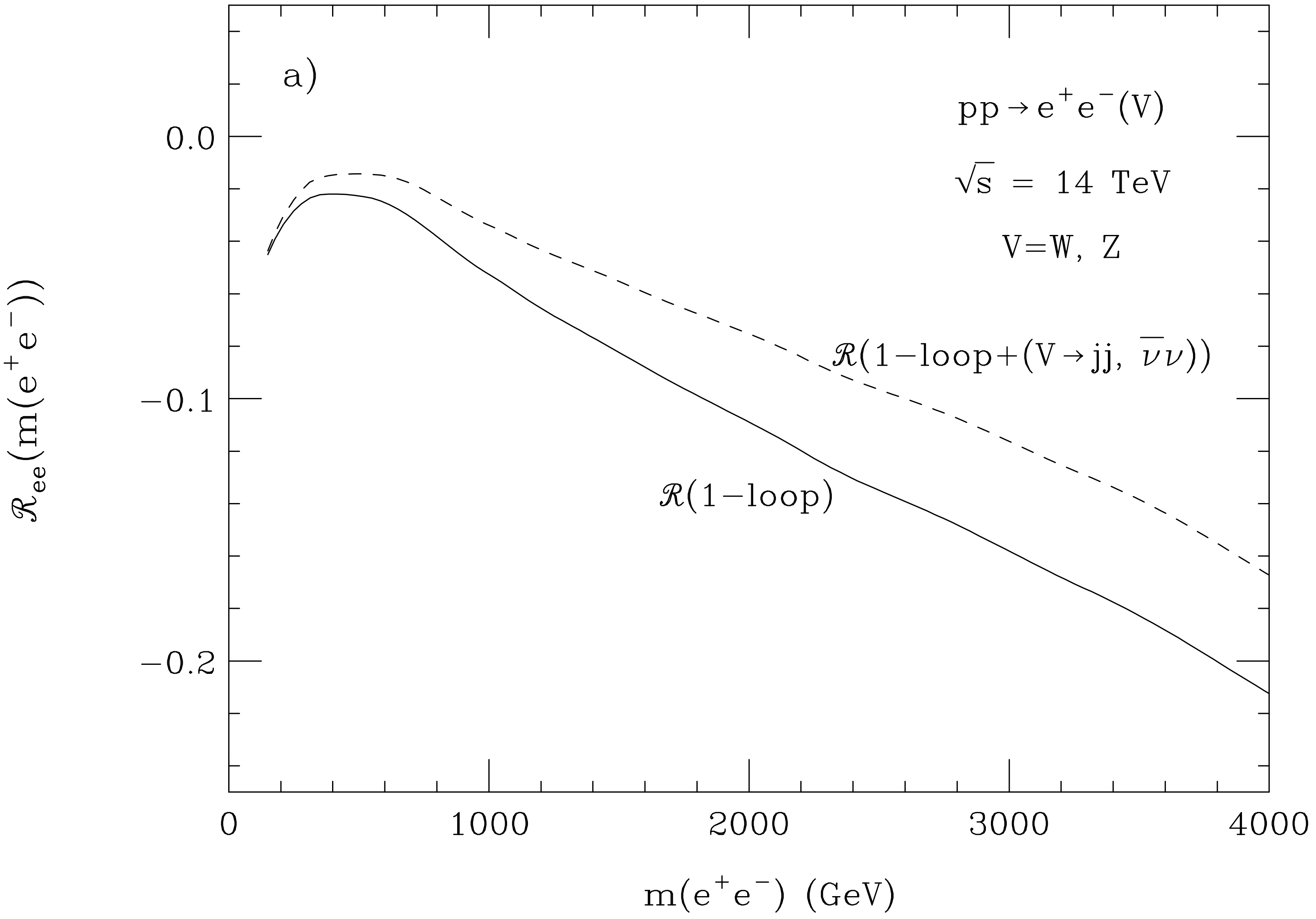
- 2-loop EW NLL corrections in $D = 4 - 2\epsilon$ dimensions
- loop integrals calculated with two independent methods, expansion by regions & Mellin–Barnes representations presented here
- Yukawa contributions only in wave-function renormalization
- universal correction factors, electromagnetic singularities separable
- process-independent: applicable for $e^+e^- \to f\bar{f}$, Drell–Yan, $gg \to f\bar{f}$, . . .
- NLL result available to all orders in ϵ → structure with powers of $(Q^2/M_\mathrm{W}^2)^\epsilon$

Outlook: EW corrections to arbitrary high-energy processes

- process-independent results for all Standard-Model particles possible at 1 loop
  Denner, Pozzorini '00, '01
- generalize 2-loop method for external gauge bosons & scalars (Higgs)
- calculate relevant loop integrals $\rightsquigarrow$ many already done for fermions

Extra slides

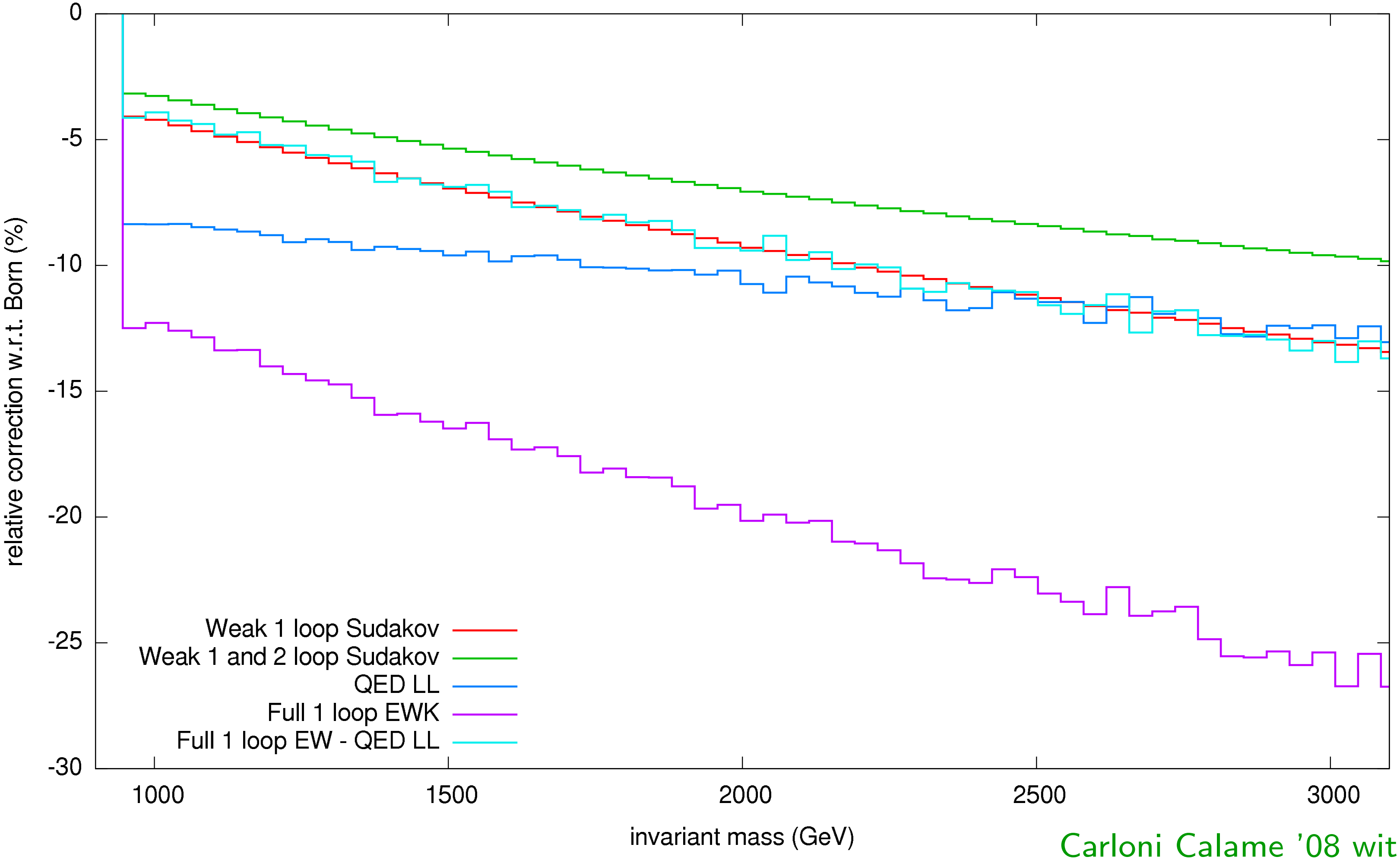
Virtual + Real W, Z emission: only partial cancellation



Baur, Phys. Rev. D75 (2007) 013005

EW 1-/2-loop corrections at the LHC

Drell–Yan $pp \to \mu^+\mu^-$: (electro)weak 1-loop & 2-loop corrections



Carloni Calame '08 with HORACE
and logarithmic ("Sudakov") results from
B.J., Kühn, Penin, Smirnov '05

$\Rightarrow$ logarithmic approximation very good at high energies

$\Rightarrow$ 2-loop effects $\sim \mathcal{O}(\%)$

cf. Les Houches 2007 report, arXiv:0803.0678 [hep-ph]

Treatment of ultraviolet (UV) singularities

UV $1/\epsilon$ poles in subdiagrams with scale μ^2_{loop} & renormalization at scale μ^2_{R}:

$$\underbrace{\frac{1}{\epsilon}\left(\frac{Q^2}{\mu^2_{\text{loop}}}\right)^{\epsilon}}_{\text{bare diagrams}} - \underbrace{\frac{1}{\epsilon}\left(\frac{Q^2}{\mu^2_{\text{R}}}\right)^{\epsilon}}_{\text{counterterms}} = \ln\left(\frac{\mu^2_{\text{R}}}{\mu^2_{\text{loop}}}\right) + \mathcal{O}(\epsilon) \quad \Rightarrow \quad \text{possible NLL contribution}$$

Perform **minimal UV subtraction** in UV-singular (sub)diagrams and counterterms:

$$\underbrace{\frac{1}{\epsilon}\left[\left(\frac{Q^2}{\mu^2_{\text{loop}}}\right)^{\epsilon} - 1\right]}_{\text{bare diagrams}} - \underbrace{\frac{1}{\epsilon}\left[\left(\frac{Q^2}{\mu^2_{\text{R}}}\right)^{\epsilon} - 1\right]}_{\text{counterterms}}$$

Advantages:

- no UV NLL terms from hard subdiagrams ($\mu^2_{\text{loop}} \sim Q^2$)
  $\hookrightarrow$ no UV contributions from internal parts of tree subdiagrams
- can use soft–collinear approximation (not valid in UV regime!) also for hard UV-singular subdiagrams

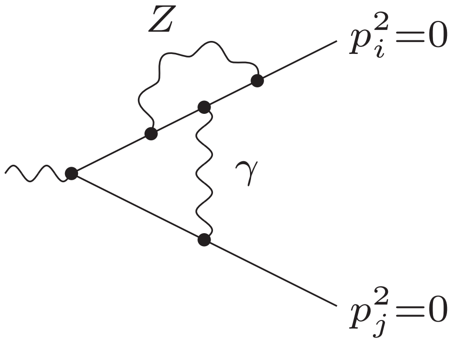
Power singularities Q^2/M^2

Asymptotic expansion for $Q^2 \gg M^2_{\mathrm{W,Z}},\, m^2_{\mathrm{t}}$

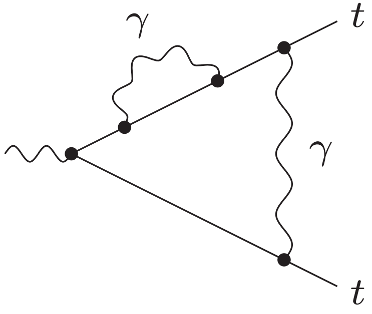
$\hookrightarrow$ logarithmic singularities $\ln(Q^2/M^2_{\mathrm{W,Z}})$, $\ln(Q^2/m^2_{\mathrm{t}})$

$\hookrightarrow$ power singularities $Q^2/M^2_{\mathrm{W,Z}}$, Q^2/m^2_{t}

E.g. in scalar diagrams (master integrals)



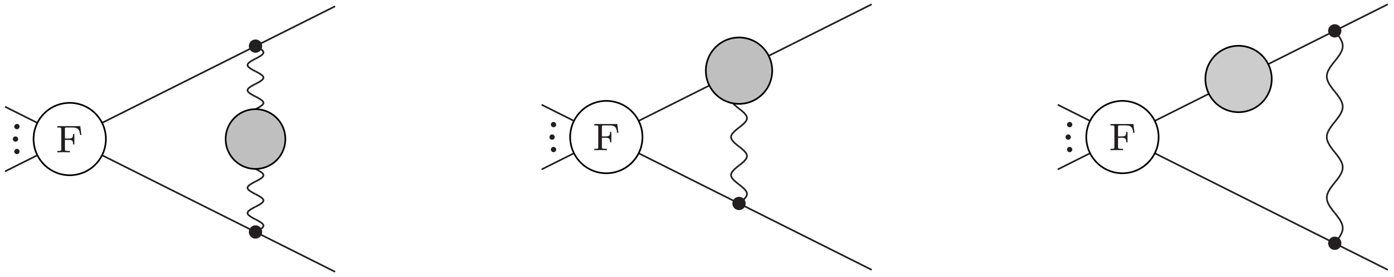
$$\overset{\text{NLL}}{=} -\frac{1}{Q^4}\,\frac{Q^2}{M_{\mathrm{Z}}^2}\left(\frac{3}{4}\epsilon^{-3} + \frac{3}{2}L\epsilon^{-2} + \frac{3}{2}L^2\epsilon^{-1} + L^3\right)$$



$$\overset{\text{NLL}}{=} -\frac{1}{Q^4}\,\frac{Q^2}{m_{\mathrm{t}}^2}\left(\frac{1}{4}L\epsilon^{-2} + \frac{3}{4}L^2\epsilon^{-1} + \frac{7}{6}L^3\right)$$

Power singularities Q^2/M^2 (2)

- not present at 1 loop
- appear at 2 loops in (scalar) diagrams with loop insertions at soft–collinear lines:



- expansion by regions predicts where power singularities can appear: simply combine the factors of M from propagators and integration measures for each region

Complete Feynman diagrams:

- Q^2/M^2 singularities always compensated by factors of M^2/Q^2 from Feynman rules or reductions ⇒ results for Feynman diagrams are free from power singularities
- **massless fermions:** power singularities do not affect fermion lines
- **massive fermions:** mass terms in numerator of fermion lines important!

  ↪ soft–collinear approximation not possible for the subdiagrams from above

  ⇒ treat these subdiagrams with projection techniques

Power singularities: complications from fermion masses

Mass terms in numerator of fermion lines important:

- fermion propagators: $\frac{\not{k} + m_\mathrm{t}}{k^2 - m_\mathrm{t}^2}$
- spinors: $(\not{p} - m_\mathrm{t})\, u(p) = 0, \quad (\not{p} + m_\mathrm{t})\, v(p) = 0$

⇒ Need more complicated projection on Dirac structure than in massless case:

$$\mathcal{M} = G_0\, \Gamma(p_i, p_j)\, u(p_i) = \Gamma_1 \underbrace{G_0\, u(p_i)}_{\text{Born}} \quad + \quad \underbrace{\Gamma_2\, G_0\, \not{p}_j\, u(p_i)}_{\text{suppressed for } Q^2 \gg m_\mathrm{t}^2}$$

⇒ Spin eigenstates $u(p)$ are not exact eigenstates of the chirality projectors $\omega_{\mathrm{R,L}} = \frac{1}{2}(1 \pm \gamma^5)$.

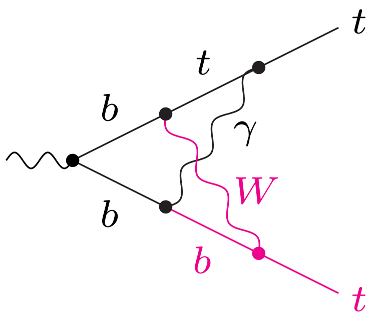
⇒ One heavy-quark line may involve EW couplings with different chiralities
↪ happens only with QED couplings which are independent of chirality.

Structure of the corrections: new logarithms

Scale $\Delta \sim M_\mathrm{W}^2 \ll Q^2$ of soft–collinear contributions $\longrightarrow$ logs

$$\ln^n\left(\frac{Q^2}{\Delta}\right) \overset{\text{NLL}}{=} \ln^n\left(\frac{Q^2}{M_\mathrm{W}^2}\right) - n\ln\left(\frac{\Delta}{M_\mathrm{W}^2}\right)\ln^{n-1}\left(\frac{Q^2}{M_\mathrm{W}^2}\right)$$

- massless fermions: only $\Delta = M_{\mathrm{Z,W}}^2$
- massive fermions: also $\Delta = m_\mathrm{t}^2$ and $\Delta = M_\mathrm{W}^2 - m_\mathrm{t}^2 - i0$ (at W–t–b vertices):



scalar diagram $\propto \frac{5}{6}L^4 - \frac{4}{3}\left[\ln\left(\frac{m_\mathrm{t}^2}{M_\mathrm{W}^2}\right) + \ln\left(\frac{M_\mathrm{W}^2 - m_\mathrm{t}^2}{M_\mathrm{W}^2}\right)\right]L^3$

Feynman diagram $\propto \frac{2}{3}L^4 - \left[4 + \frac{2}{3}\ln\left(\frac{m_\mathrm{t}^2}{M_\mathrm{W}^2}\right)\right]L^3$

Complete Feynman diagrams:

- diagrams **without photons:** independent of fermion masses, only $\Delta = M_{\mathrm{Z,W}}^2$
- diagrams **with photons:** $\Delta = M_{\mathrm{Z,W}}^2$ and $\Delta = m_\mathrm{t}^2$, but not $\Delta = M_\mathrm{W}^2 - m_\mathrm{t}^2$
  (additionally $1/\epsilon$ poles)

Parameterization of Feynman integrals

- Schwinger parameters:

$$\frac{1}{A^n} = \frac{1}{i^n\,\Gamma(n)} \int_0^\infty \mathrm{d}\alpha\,\alpha^{n-1}\,e^{i\alpha A}\,, \quad \text{numerator } A^n = \left(\frac{1}{i}\frac{\partial}{\partial\alpha}\right)^n e^{i\alpha A}\Big|_{\alpha=0}$$

⇒ any number of propagators and numerators may be combined

⇒ can always be transformed to (generalized) Feynman parameters

↪ evaluation:

$$\int \frac{\mathrm{d}^D k}{i\pi^{D/2}}\,e^{i(\alpha k^2+2p\cdot k)} = (i\alpha)^{-D/2}\,e^{-ip^2/\alpha}$$

$$\int_0^\infty \frac{\mathrm{d}\alpha\,\alpha^{n-1}}{(A+\alpha B)^r} = \frac{\Gamma(n)\,\Gamma(r-n)}{\Gamma(r)\,A^{r-n}\,B^n}$$

- generalized Feynman parameters:

$$\prod_{i=1}^{L}\frac{1}{A_i^{n_i}} = \frac{\Gamma(\sum_i n_i)}{\prod_i \Gamma(n_i)}\left(\prod_i \int_0^\infty \mathrm{d}x_i\,x_i^{n_i-1}\right)\frac{\delta\Big(\sum\limits_{j\in S} x_j - 1\Big)}{(\sum_i x_i A_i)^{\sum_i n_i}}\,, \qquad \emptyset \neq S \subseteq \{1,\dots,L\}$$

⇒ convenient also for non-standard propagators, e.g. $A_i = 2p\cdot k$

Mellin–Barnes integrals: extraction of singularities

$$I = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \underbrace{\Gamma(\epsilon + z)}_{\text{left poles}} \underbrace{\Gamma(-z)}_{\text{right poles}} f(z)$$

⇒ The left pole at $z = -\epsilon$ and the right pole at $z = 0$ "glue together" for $\epsilon \to 0$.

Close contour to the right:

$$-\mathrm{Res}\,\Gamma(\epsilon + z)\Gamma(-z) f(z)\Big|_{z=0} = \Gamma(\epsilon) f(0) = \frac{1}{\epsilon} f(0) + \mathcal{O}(\epsilon^0)$$

⇒ When a left pole and a right pole glue together, a singularity is produced!

Extract such singularities by shifting the contour:

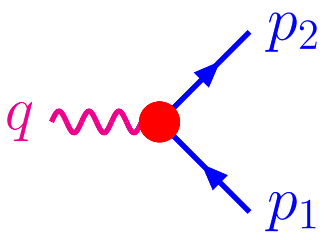
$$I = -\mathrm{Res}\,\Gamma(\epsilon + z)\Gamma(-z) f(z)\Big|_{z=0} + \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \underbrace{\frac{\Gamma(\epsilon + z)}{(-z)}}_{\text{left poles}} \underbrace{\Gamma(1 - z)}_{\text{right poles}} f(z)$$

Now the poles at $z = -\epsilon$ and $z = 0$ both lie to the left of the integration contour.

↪ The integrand can safely be expanded in ϵ.

[ϵ can be a combination of $\epsilon = (4 - D)/2$ and analytic regularization parameters.]

Expansion by regions: example



Vertex form factor in the Sudakov limit $Q^2 \gg M^2$

(massless fermions, gauge boson mass M)

- typical regions for each loop momentum k:

hard	(h):	all components of $k \sim Q$
soft	(s):	all components of $k \sim M$
ultrasoft	(us):	all components of $k \sim M^2/Q$
1-collinear	(1c):	$k^2 \sim 2p_1 \cdot k \sim M^2, \quad 2p_2 \cdot k \sim Q^2$
2-collinear	(2c):	$k^2 \sim 2p_2 \cdot k \sim M^2, \quad 2p_1 \cdot k \sim Q^2$

- 1-loop vertex correction: $f = \frac{e^{\epsilon\gamma_E}}{i\pi^{D/2}} \int \frac{\mathrm{d}^D k}{(k^2 - M^2)(k^2 - 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)}$

$$f^{(h)} = \frac{1}{Q^2}\left[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln(Q^2) - \frac{1}{2}\ln^2(Q^2) + \frac{\pi^2}{12} + \mathcal{O}\left(\frac{M^2}{Q^2}\right)\right]$$

$$f^{(1c)} + f^{(2c)} = \frac{1}{Q^2}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln(Q^2) - \frac{1}{2}\ln^2(M^2) + \ln(M^2)\ln(Q^2) - \frac{5}{12}\pi^2 + \mathcal{O}\left(\frac{M^2}{Q^2}\right)\right]$$

$$\Rightarrow f = f^{(h)} + f^{(1c)} + f^{(2c)} = \frac{1}{Q^2}\left[-\frac{1}{2}\ln^2\left(\frac{Q^2}{M^2}\right) - \frac{\pi^2}{3} + \mathcal{O}\left(\frac{M^2}{Q^2}\right)\right]$$

Expansion by regions: how its works

simple 1-dimensional example: $f = \displaystyle\int_0^\infty \frac{\mathrm{d}k\, k^{-\epsilon}}{(k+m)(k+q)}\,, \quad m \ll q$

$$\left.\begin{array}{lll} \text{soft (s):} & k \sim m, & k < \Lambda \\ \text{hard (h):} & k \sim q, & k > \Lambda \end{array}\right\} \text{where } m \ll \Lambda \ll q$$

$$f = \int_0^\Lambda \frac{\mathrm{d}k\, k^{-\epsilon}}{(k+m)(k+q)} + \int_\Lambda^\infty \frac{\mathrm{d}k\, k^{-\epsilon}}{(k+m)(k+q)}$$

$$= \sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \int_0^\Lambda \frac{\mathrm{d}k\, k^{-\epsilon+j}}{k+m} + \sum_{i=0}^\infty (-m)^i \int_\Lambda^\infty \frac{\mathrm{d}k\, k^{-\epsilon-i-1}}{k+q}$$

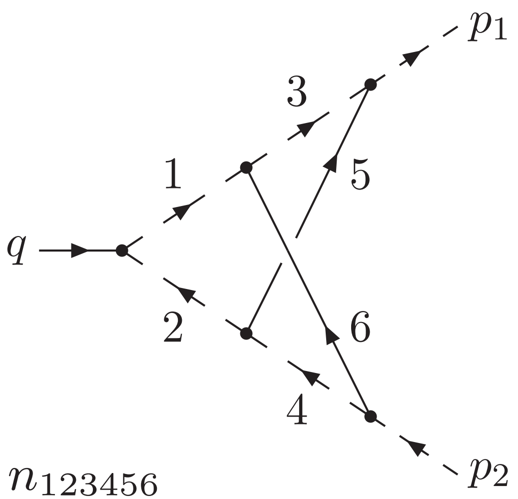
$$= \sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \left(\int_0^\infty \frac{\mathrm{d}k\, k^{-\epsilon+j}}{k+m} - \int_\Lambda^\infty \frac{\mathrm{d}k\, k^{-\epsilon+j}}{k+m} \right) + \sum_{i=0}^\infty (-m)^i \left(\int_0^\infty \frac{\mathrm{d}k\, k^{-\epsilon-i-1}}{k+q} - \int_0^\Lambda \frac{\mathrm{d}k\, k^{-\epsilon-i-1}}{k+q} \right)$$

$$= \underbrace{\sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \int_0^\infty \frac{\mathrm{d}k\, k^{-\epsilon+j}}{k+m}}_{f^{(s)}} + \underbrace{\sum_{i=0}^\infty (-m)^i \int_0^\infty \frac{\mathrm{d}k\, k^{-\epsilon-i-1}}{k+q}}_{f^{(h)}} - \sum_{i=0}^\infty (-m)^i \sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \underbrace{\int_0^\infty \mathrm{d}k\, k^{-\epsilon-i+j-1}}_{\to 0,\ \text{scaleless integral}}$$

$$= f^{(s)} + f^{(h)} = \frac{\Gamma(\epsilon)\Gamma(1-\epsilon)}{m^\epsilon\, q} \sum_{j=0}^\infty \left(\frac{m}{q}\right)^j + \frac{\Gamma(-\epsilon)\Gamma(1+\epsilon)}{q^{1+\epsilon}} \sum_{i=0}^\infty \left(\frac{m}{q}\right)^i$$

$$= \frac{\Gamma(\epsilon)\Gamma(1-\epsilon)}{(q-m)\, m^\epsilon} + \frac{\Gamma(-\epsilon)\Gamma(1+\epsilon)}{(q-m)\, q^\epsilon} = \frac{\ln(q/m)}{q-m} + \mathcal{O}(\epsilon) \quad \checkmark$$

Example: the non-planar vertex diagram



Scalar integrals with variable powers of propagators:

$$F_{\text{NP}}(n_1,\ldots,n_7) = e^{2\epsilon\gamma}\,(M^2)^{2\epsilon}\,(Q^2)^{n-n_7-4}$$

$$\times \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \int \frac{\mathrm{d}^D \ell}{i\pi^{D/2}} \, \frac{(2k\cdot\ell)^{n_7}}{\big((p_1-k-\ell)^2\big)^{n_1}\,\big((p_2-k-\ell)^2\big)^{n_2}}$$

$$\times \frac{1}{(k^2-2p_1\cdot k)^{n_3}\,(\ell^2-2p_2\cdot\ell)^{n_4}\,(k^2-M^2)^{n_5}\,(\ell^2-M^2)^{n_6}}\,, \qquad n = n_{123456}$$

Contributing regions: (h-h), (1c-h), (h-2c), (1c-1c), (2c-2c), (1c-2c), (1c-1c'), (2c'-2c), (us'-us'), (1c-us'), (us'-2c).

Leading term of (1c-h) region $\iff$ $k^2 \sim 2p_1\cdot k \sim M^2$, $2p_2\cdot k \sim Q^2$, $\ell \sim Q$:

$$F_{\text{NP}}^{(\text{1c-h})}(n_1,\ldots,n_7) = e^{2\epsilon\gamma}\,(M^2)^{2\epsilon}\,(Q^2)^{n-n_7-4} \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \int \frac{\mathrm{d}^D \ell}{i\pi^{D/2}}$$

$$\times \frac{\big((2p_2\cdot k)(2p_1\cdot\ell)/Q^2\big)^{n_7}}{\big(\ell^2-2p_1\cdot\ell+(2p_2\cdot k)(2p_1\cdot\ell)/Q^2\big)^{n_1}\,\big(\ell^2-2p_2\cdot(k+\ell)+(2p_2\cdot k)(2p_1\cdot\ell)/Q^2\big)^{n_2}}$$

$$\times \frac{1}{(k^2-2p_1\cdot k)^{n_3}\,(\ell^2-2p_2\cdot\ell)^{n_4}\,(k^2-M^2)^{n_5}\,(\ell^2)^{n_6}} + \mathcal{O}\left(\frac{M^2}{Q^2}\right)$$

Example: (1c-h) region of the non-planar vertex diagram

Introduce Feynman or Schwinger parameters, integrate & transform into

$$F_{\mathrm{NP}}^{(\text{1c-h})}(n_1,\ldots,n_7) = \left(\frac{M^2}{Q^2}\right)^{2-n_{35}+\epsilon} (-1)^n \, e^{2\epsilon\gamma} \, \frac{\Gamma(\frac{D}{2}-n_{24})\Gamma(\frac{D}{2}-n_{16}+n_7)\Gamma(n_{35}-\frac{D}{2})}{\Gamma(D-n_{1246}+n_7)\,\prod_{i=1}^{6}\Gamma(n_i)}$$

$$\times \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, x_1^{n_1-1}(1-x_1)^{n_6-1} x_2^{n_2-1}(1-x_2)^{n_4-1} x_3^{n_{37}-1}(1-x_3)^{\frac{D}{2}-n_3-1}$$

$$\times \underbrace{\Gamma(n_{1246}-\tfrac{D}{2})\,\big[x_1(1-x_3)+x_2 x_3\big]^{\frac{D}{2}-n_{1246}}}_{\text{Mellin–Barnes representation:}}$$

$$\int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i}\, \Gamma(-z)\Gamma(n_{1246}-\tfrac{D}{2}+z)\, \big(x_1(1-x_3)\big)^z \, (x_2 x_3)^{\frac{D}{2}-n_{1246}-z}$$

⇒ Expression with Mellin–Barnes integral:

$$F_{\mathrm{NP}}^{(\text{1c-h})}(n_1,\ldots,n_7) = \left(\frac{M^2}{Q^2}\right)^{2-n_{35}+\epsilon} (-1)^n \, \frac{e^{2\epsilon\gamma}\,\Gamma(\frac{D}{2}-n_{24})\Gamma(\frac{D}{2}-n_{16}+n_7)\Gamma(n_{35}-\frac{D}{2})}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)\Gamma(n_5)\,\Gamma^2(D-n_{1246}+n_7)}$$

$$\times \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \, \frac{\Gamma(-z)\Gamma(\frac{D}{2}-n_{146}-z)\Gamma(\frac{D}{2}-n_{1246}+n_{37}-z)}{\Gamma(\frac{D}{2}-n_{16}-z)}$$

$$\times \frac{\Gamma(n_1+z)\Gamma(\frac{D}{2}-n_3+z)\Gamma(n_{1246}-\frac{D}{2}+z)}{\Gamma(n_{16}+z)}$$

Example: evaluation of the (1c-h) region for special cases (1)

$$F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,1,1,0,1,0,0) = \left(\frac{M^2}{Q^2}\right)^{\epsilon} e^{2\epsilon\gamma}\,\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)^2} \times \int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2\pi i}\,\Gamma(-z)\Gamma(1-\epsilon-z)\,\Gamma(1-\epsilon+z)\Gamma(\epsilon+z)$$

Solution known: 1st Barnes lemma

Barnes 1908

$$\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2\pi i}\,\Gamma(\alpha_1-z)\Gamma(\alpha_2-z)\,\Gamma(\alpha_3+z)\Gamma(\alpha_4+z) = \frac{\Gamma(\alpha_1+\alpha_3)\Gamma(\alpha_1+\alpha_4)\Gamma(\alpha_2+\alpha_3)\Gamma(\alpha_2+\alpha_4)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$$

$$\Rightarrow F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,1,1,0,1,0,0) = \left(\frac{M^2}{Q^2}\right)^{\epsilon} e^{2\epsilon\gamma}\,\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)^2}\,\frac{\Gamma(1-\epsilon)\Gamma(\epsilon)\Gamma(2-2\epsilon)}{\Gamma(2-\epsilon)}$$

$$= \frac{1}{\epsilon^2}+\frac{1}{\epsilon}\Big(-L+3\Big)+\frac{1}{2}L^2-3L+7+\mathcal{O}(\epsilon)\,, \quad L=\ln\!\left(\frac{Q^2}{M^2}\right)$$

Example: evaluation of the (1c-h) region for special cases (2)

$$F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,1,1,0,1,1,0) = -\left(\frac{M^2}{Q^2}\right)^{\epsilon} e^{2\epsilon\gamma}\,\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(\epsilon)}{\Gamma(1-2\epsilon)^2} \times \int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2\pi i}\,\Gamma(-z)\Gamma(-\epsilon-z)\,\frac{\Gamma(1+z)\Gamma(1-\epsilon+z)\Gamma(1+\epsilon+z)}{\Gamma(2+z)}$$

- 1st possibility: expand integrand in ϵ and/or cancel functions in denominator
  $\hookrightarrow$ transform to expressions solvable by Barnes lemma etc.
- 2nd possibility: close integration contour to the right and take residues directly:

$$\left(\frac{M^2}{Q^2}\right)^{\epsilon} e^{2\epsilon\gamma}\,\frac{\Gamma(1-\epsilon)^3\Gamma(1+\epsilon)^2}{\epsilon^3\,\Gamma(1-2\epsilon)^2}\sum_{i=0}^{\infty}\left[\underbrace{\frac{\Gamma(1-2\epsilon+i)}{\Gamma(2-\epsilon+i)}}_{\text{from } z=-\epsilon+i} - \underbrace{\frac{\Gamma(1-\epsilon+i)}{\Gamma(2+i)}}_{\text{from } z=i}\right]$$

expand Gamma functions → sum up to (multiple) zeta values:

$$F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,1,1,0,1,1,0) = \frac{\pi^2}{6\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{\pi^2}{6}L + 2\zeta_3\right) + \frac{\pi^2}{12}L^2 - 2\zeta_3 L + \frac{\pi^4}{40} + \mathcal{O}(\epsilon)$$

Example: evaluation of the (1c-h) region for special cases (3)

$$F_{\text{NP}}^{(\text{1c-h})}(1,\delta,1,1,1,1,0) = -\left(\frac{M^2}{Q^2}\right)^{\epsilon} (-1)^{\delta}\, e^{2\epsilon\gamma}\, \frac{\Gamma(1-\epsilon-\delta)\Gamma(-\epsilon)\Gamma(\epsilon)}{\Gamma(\delta)\Gamma(1-2\epsilon-\delta)^2}$$

$$\times \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i}\, \frac{\Gamma(-z)\Gamma(-1-\epsilon-z)\Gamma(-\epsilon-\delta-z)}{\Gamma(-\epsilon-z)}\, \frac{\Gamma(1+z)\Gamma(1-\epsilon+z)\Gamma(1+\epsilon+\delta+z)}{\Gamma(2+z)}$$

Limit $\delta \to 0 \Rightarrow \frac{1}{\Gamma(\delta)} \to 0$, but gluing poles at $z = -1-\epsilon$ and $z = -1-\epsilon-\delta$:

$$F_{\text{NP}}^{(\text{1c-h})}(1,0,1,1,1,1,0) = \lim_{\delta\to 0} \left(\frac{M^2}{Q^2}\right)^{\epsilon} (-1)^{\delta}\, e^{2\epsilon\gamma}\, \frac{\Gamma(1-\epsilon-\delta)\Gamma(-\epsilon)\Gamma(\epsilon)}{\Gamma(\delta)\Gamma(1-2\epsilon-\delta)^2}$$

$$\times \operatorname{Res} \left. \frac{\Gamma(-z)\Gamma(-1-\epsilon-z)\Gamma(-\epsilon-\delta-z)}{\Gamma(-\epsilon-z)}\, \frac{\Gamma(1+z)\Gamma(1-\epsilon+z)\Gamma(1+\epsilon+\delta+z)}{\Gamma(2+z)} \right|_{z=-1-\epsilon}$$

$$= -\left(\frac{M^2}{Q^2}\right)^{\epsilon} e^{2\epsilon\gamma}\, \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(\epsilon)}{\Gamma(1-2\epsilon)^2}\, \frac{\Gamma(1+\epsilon)\Gamma(-\epsilon)\Gamma(-2\epsilon)}{\Gamma(1-\epsilon)}$$

$$= \frac{1}{2\epsilon^4} - \frac{1}{2\epsilon^3}L + \frac{1}{4\epsilon^2}L^2 - \frac{1}{\epsilon}\left(\frac{1}{12}L^3 + \frac{4}{3}\zeta_3\right) + \frac{1}{48}L^4 + \frac{4}{3}\zeta_3 L - \frac{\pi^4}{60} + \mathcal{O}(\epsilon)$$