Theorie-Palaver
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Asymptotic expansions with the strategy of regions

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Overview

I The strategy of regions

- asymptotic expansion of loop integrals
- introduction to the expansion by regions
- example: large-momentum expansion

II Why does the method work?

- transforming original integral $\to$ series of expanded integrals
- overlap contribution

III Examples

- threshold expansion for heavy-particle pair production
- Sudakov form factor ($\rightsquigarrow$ non-commuting expansions)

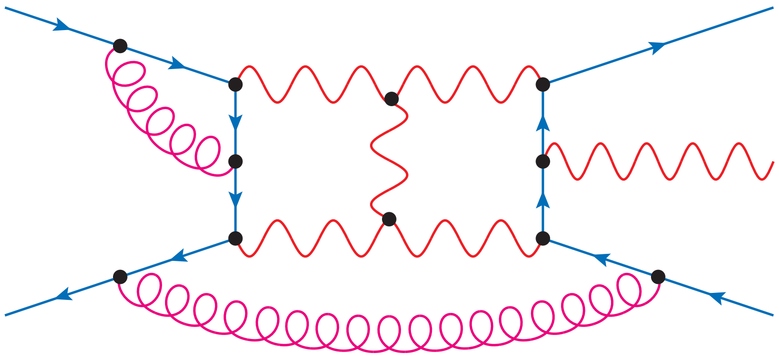
IV The general formalism

- conditions on regions & expansions
- general identity with overlap contributions
- example: forward scattering with small momentum exchange

V Summary

Overview

I The strategy of regions



Starting point: (multi-)loop integral

(or other complicated integral)

$$F = \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \cdots I \,,$$

$$I = \frac{1}{(k_1+p_1)^2 - m_1^2} \, \frac{1}{(k_1+k_2+p_2)^2 - m_2^2} \cdots$$

- complicated function of internal masses m_i and kinematical parameters p_i^2, $p_i \cdot p_j$
- exact evaluation often hard or impossible

Exploit parameter hierarchies, e.g. large energies $Q \gg$ small masses m:

- expand integral in small ratios $\frac{m^2}{Q^2}$: $F = F_0 + \frac{m^2}{Q^2} F_1 + \left(\frac{m^2}{Q^2}\right)^2 F_2 + \ldots$
- simplification achieved if expansion of integrand before integration:
  $I \to I_0 + \frac{m^2}{Q^2} I_1 + \left(\frac{m^2}{Q^2}\right)^2 I_2 + \ldots \,, \qquad F_j = \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \cdots I_j$
- expanded integrands I_j often simpler to integrate than original integrand I

Expansion of integrand before integration?

$$I \to I_0 + \frac{m^2}{Q^2} I_1 + \left(\frac{m^2}{Q^2}\right)^2 I_2 + \ldots, \qquad F_j = \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \cdots I_j$$

But:

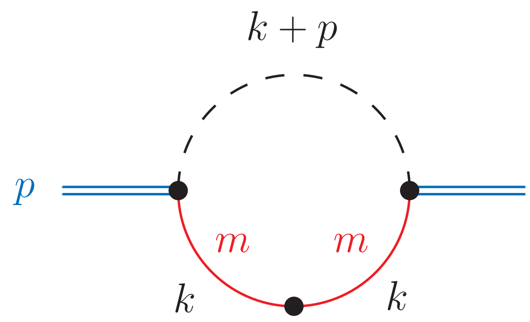
- ⋆ integrand I is function of loop momenta: $I = I(k_1, k_2, \ldots)$
- ⋆ loop-momentum components k_i^μ can take any values (large, small, mixed, ...)
- ⋆ expansions of integrand may break down for certain values of $k_1, k_2, \ldots$
- ⋆ naive integrations of expanded integrand may generate new singularities

$\hookrightarrow$ Need sophisticated methods of asymptotic expansions.

Simple example: large-momentum expansion

$$F = \int \frac{\mathrm{D}k}{(k+p)^2\,(k^2-m^2)^2}$$

$$\left[\int \mathrm{D}k \equiv \mu^{2\epsilon} e^{\epsilon\gamma_E} \int \frac{\mathrm{d}^d k}{i\pi^{d/2}} \qquad d = 4-2\epsilon \right]$$



Large momentum $|p^2| \gg m^2$ $\rightsquigarrow$ expand in $\frac{m^2}{p^2}$.

Integral is UV- and IR-finite, the exact result is known: $[p^2 \to p^2 + i0]$

$$F = \frac{1}{p^2}\left[\ln\left(\frac{-p^2}{m^2}\right) + \ln\left(1-\frac{m^2}{p^2}\right)\right] + \mathcal{O}(\epsilon)$$

$$\xrightarrow[\text{expand}]{} \frac{1}{p^2}\left[\ln\left(\frac{-p^2}{m^2}\right) - \sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{m^2}{p^2}\right)^j\right] + \mathcal{O}(\epsilon)$$

[Appearance of logarithm $\rightsquigarrow$ simple expansion of integrand in powers of m^2 is incorrect!]

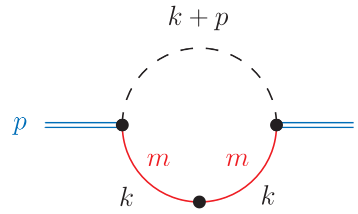
Now assume that we could <u>not</u> calculate this integral exactly . . .

Large-momentum expansion (2)

Large momentum $|p^2| \gg m^2$

$\hookrightarrow$ expand integrand before integration:

$$F = \int \frac{\mathrm{D}k}{(k+p)^2\,(k^2-m^2)^2}$$



Expansion by regions

Beneke, Smirnov, Nucl. Phys. B 522 (1998) 321
Smirnov, Rakhmetov, Theor. Math. Phys. 120 (1999) 870
Smirnov, Phys. Lett. B 465 (1999) 226

$\hookrightarrow$ here 2 relevant **regions**:

- **hard** (h): $k \sim p \Rightarrow \dfrac{1}{(k+p)^2\,(k^2-m^2)^2} \to \dfrac{1}{(k+p)^2}\left(\dfrac{1}{(k^2)^2} + \dfrac{2m^2}{(k^2)^3} + \ldots\right)$
- **soft** (s): $k \sim m \Rightarrow \dfrac{1}{(k+p)^2\,(k^2-m^2)^2} \to \dfrac{1}{(k^2-m^2)^2}\left(\dfrac{1}{p^2} - \dfrac{2k\cdot p}{(p^2)^2} - \dfrac{k^2}{(p^2)^2} + \ldots\right)$

$\Rightarrow$ Integrate each expanded term over the **whole integration domain**.

$\Rightarrow$ Set scaleless integrals to zero (like in dimensional regularization).

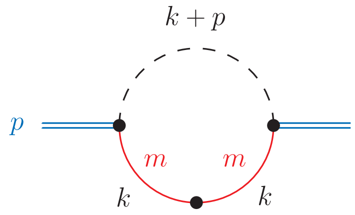
Leading-order contributions:

- **hard:** $F_0^{(h)} = \displaystyle\int \frac{\mathrm{D}k}{(k+p)^2\,(k^2)^2} = \frac{1}{p^2}\left(\frac{\mu^2}{-p^2}\right)^{\epsilon}\left(-\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right)$
- **soft:** $F_0^{(s)} = \displaystyle\int \frac{\mathrm{D}k}{p^2\,(k^2-m^2)^2} = \frac{1}{p^2}\left(\frac{\mu^2}{-p^2}\right)^{\epsilon}\left(\frac{m^2}{-p^2}\right)^{-\epsilon}\left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right)$

$\hookrightarrow$ Contributions are homogeneous functions of the expansion parameter $\frac{m^2}{p^2}$.

Large-momentum expansion (3)

$$F = \int \frac{\mathrm{D}k}{(k+p)^2\,(k^2-m^2)^2}$$



Leading-order contributions:

- **hard:** $F_0^{(h)} = \int \frac{\mathrm{D}k}{(k+p)^2\,(k^2)^2} = \frac{1}{p^2}\left[-\frac{1}{\epsilon} + \ln\left(\frac{-p^2}{\mu^2}\right)\right] + \mathcal{O}(\epsilon) \rightsquigarrow$ IR-singular!
- **soft:** $F_0^{(s)} = \frac{1}{p^2}\int \frac{\mathrm{D}k}{(k^2-m^2)^2} = \frac{1}{p^2}\left[\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right)\right] + \mathcal{O}(\epsilon) \rightsquigarrow$ UV-singular!

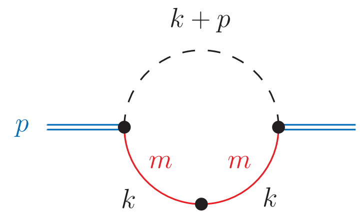
↪ Singularities are cancelled in the sum of all contributions.

↪ Exact result is approximated:

$$F_0 = F_0^{(h)} + F_0^{(s)} = \frac{1}{p^2}\ln\left(\frac{-p^2}{m^2}\right) + \mathcal{O}(\epsilon) = F + \mathcal{O}\left(\frac{m^2}{(p^2)^2}\right) \quad \checkmark$$

Large-momentum expansion (4)

$$F = \int \frac{\mathrm{D}k}{(k+p)^2\,(k^2-m^2)^2}$$



Expansion to all orders in $\frac{m^2}{p^2}$:

$[(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)]$

- **hard:** $\sum_i T_i^{(h)} \frac{1}{(k^2-m^2)^2} = \sum_{i=0}^{\infty} (1+i) \frac{(m^2)^i}{(k^2)^{2+i}}$

$$\hookrightarrow \quad F^{(h)} = \frac{1}{p^2} \left(\frac{\mu^2}{-p^2}\right)^{\epsilon} \frac{e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)\,\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i=0}^{\infty} \left(\frac{m^2}{p^2}\right)^i \frac{(2\epsilon)_i}{i!}$$

$$= \frac{1}{p^2} \left[-\frac{1}{\epsilon} + \ln\left(\frac{-p^2}{\mu^2}\right) + 2\ln\left(1-\frac{m^2}{p^2}\right) \right] + \mathcal{O}(\epsilon)$$

- **soft:** $\sum_j T_j^{(s)} \frac{1}{(k+p)^2} = \sum_{j_1,j_2=0}^{\infty} \frac{(j_1+j_2)!}{j_1!\,j_2!} \frac{(-2k\cdot p)^{j_1}\,(-k^2)^{j_2}}{(p^2)^{1+j_1+j_2}}$

$$\hookrightarrow \quad F^{(s)} = \frac{1}{p^2} \left(\frac{\mu^2}{m^2}\right)^{\epsilon} e^{\epsilon\gamma_E}\,\Gamma(\epsilon) \sum_{j=0}^{\infty} \left(\frac{m^2}{p^2}\right)^j \frac{(\epsilon)_j}{(1-\epsilon)_j}$$

$$= \frac{1}{p^2} \left[\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right) - \ln\left(1-\frac{m^2}{p^2}\right) \right] + \mathcal{O}(\epsilon)$$

Full result F exactly reproduced:

$$F = F^{(h)} + F^{(s)} = \frac{1}{p^2} \left[\ln\left(\frac{-p^2}{m^2}\right) + \ln\left(1-\frac{m^2}{p^2}\right) \right] + \mathcal{O}(\epsilon) \quad \checkmark$$

Questions: Why does this expansion by regions work?

- What ensures the cancellation of singularities? (IR $\leftrightarrow$ UV!)

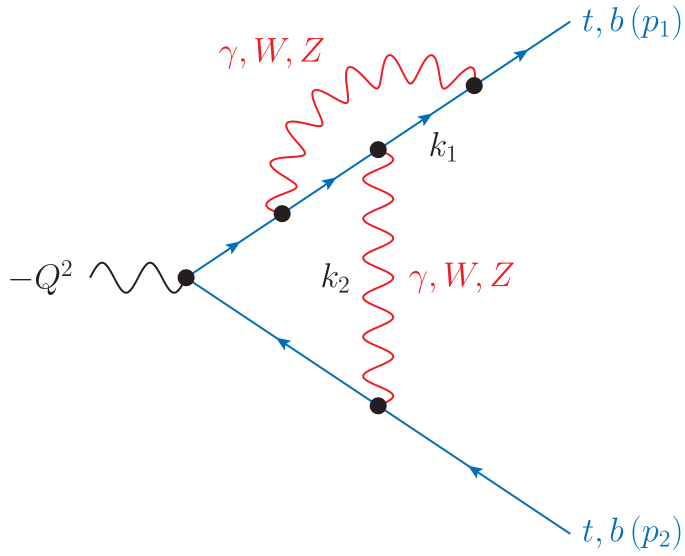
- Didn't we double-count every $k \in \mathbb{R}^d$ when replacing (for the leading order) $\int \mathrm{D}k \to \int \mathrm{D}k \, T_0^{(h)} + \int \mathrm{D}k \, T_0^{(s)}$?

- How do we have to choose the regions?
  And how do we know that the chosen set of regions is complete?

- What is the role of scaleless integrals?

The expansion by regions has been applied successfully to many complicated loop integrals.

"Real-life" example



2-loop vertex integral in the high-energy limit

Denner, B.J., Pozzorini '08

$$Q^2 \gg m_t^2 \sim M_{W,Z}^2$$

↪ 9 relevant regions: [labelled "$(k_1 - k_2)$"]

$(h-h)$, $(1c-h)$, $(h-2c)$,
$(1c-1c)$, $(1c-2c)$, $(2c-2c)$,
$(1c-2uc)$, $(2uc-2uc)$, $(us-2c)$

- next-to-leading-logarithmic result obtained: $\alpha^2 \left\{L^3,\, L^2/\epsilon,\, L/\epsilon^2,\, 1/\epsilon^3\right\}$, where $L = \ln(Q^2/M_W^2)$
- cross-checked with independent calculation based on sector decomposition

Practical note: how to find the relevant regions

- Look where the propagators have poles:
  - ⋆ Large-momentum example: $(k+p)^2 = 0$ at $k \sim p$, $k^2 - m^2 = 0$ at $k \sim m$.
  - ⋆ Close the integration contour of one component (e.g. k^0, $k^\pm$). For all residues investigate the scaling of the components.

- Use Mellin–Barnes (MB) representations:

$$\boxed{\frac{1}{(A+B)^n} = \frac{1}{\Gamma(n)} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2i\pi}\, \Gamma(n+z)\, \Gamma(-z)\, \frac{B^z}{A^{n+z}}}$$

  1. Evaluate the full (scalar) integral for generic propagator powers n_i in terms of multiple MB integrals.
  2. Close MB contours involving the expansion parameter and extract the leading contributions.
  3. The individual terms can be identified with corresponding regions by their homogeneous scaling with the expansion parameter depending on d and n_i.

  [A subsequent expansion by regions often yields simpler expressions for the contributions.]

Practical note: how to find the relevant regions (2)

- Try all possible regions that you can imagine ...
  If a region does not contribute, its integrals are scaleless.

- Automated by Mathematica code `asy.m`, Pak, A. Smirnov, Eur. Phys. J. C 71 (2011) 1626
  finds non-scaleless contributions automatically via geometric approach:

  ```
  AlphaRepExpand[{k}, {(k+p)^2, k^2-m^2}, {p^2->1}, {m^2->x}]
  ```

  Expansion based on Feynman-parameter integral $\rightsquigarrow$ result: list of regions
  with scalings of Feynman parameters in powers of the expansion parameter

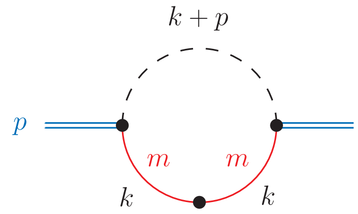
  Published version of `asy.m`: potential & Glauber regions not found
  $\hookrightarrow$ update available soon B.J., A. Smirnov, V. Smirnov, *to be published*

- When a region is missing, the total result is often (but not always) more singular than it should be. $\rightsquigarrow$ Important cross-check, but no guarantee!

Overview

II Why does the method work?

Idea based on a 1-dimensional toy example from M. Beneke (1997)
[see also: Smirnov, *Applied Asymptotic Expansions In Momenta And Masses*]



Large-momentum example

Let us show step by step how the expansions reproduce the full result.

The **expansions** $\sum_i T_i^{(h)}$, $\sum_j T_j^{(s)}$ **converge absolutely** within **domains** D_h, D_s:

$$(h):\ \frac{1}{(k^2-m^2)^2} = \sum_i T_i^{(h)} \frac{1}{(k^2-m^2)^2} \text{ within } D_h = \left\{ k \in \mathbb{R}^d : |k^2| \geq \Lambda^2 \right\},$$

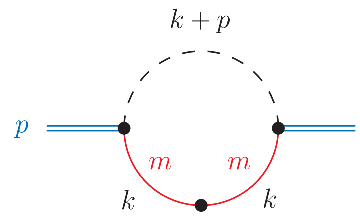
$$(s):\ \frac{1}{(k+p)^2} = \sum_j T_j^{(s)} \frac{1}{(k+p)^2} \text{ within } D_s = \left\{ k \in \mathbb{R}^d : |k^2| < \Lambda^2 \right\},$$

with $m^2 \ll \Lambda^2 \ll |p^2| \rightsquigarrow D_h \cup D_s = \mathbb{R}^d,\ D_h \cap D_s = \emptyset$.

The expansions **commute** with integrals restricted to the corresponding domains:

$$\int\limits_{k\in D_h} \mathrm{D}k \underbrace{\frac{1}{(k+p)^2\,(k^2-m^2)^2}}_{I} = \sum_i \int\limits_{k\in D_h} \mathrm{D}k\, T_i^{(h)} I\,, \qquad \int\limits_{k\in D_s} \mathrm{D}k\, I = \sum_j \int\limits_{k\in D_s} \mathrm{D}k\, T_j^{(s)} I$$

Transform the expression for the full integral:



$$F = \int\limits_{k\in D_h} \mathrm{D}k\, I + \int\limits_{k\in D_s} \mathrm{D}k\, I = \sum_i \int\limits_{k\in D_h} \mathrm{D}k\, T_i^{(h)} I + \sum_j \int\limits_{k\in D_s} \mathrm{D}k\, T_j^{(s)} I$$

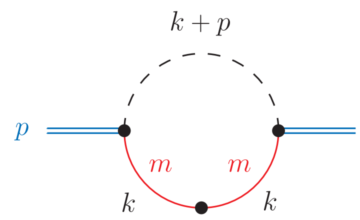
$$= \sum_i \left(\int\limits_{k\in\mathbb{R}^d} \mathrm{D}k\, T_i^{(h)} I - \sum_j \int\limits_{k\in D_s} \mathrm{D}k\, T_j^{(s)} T_i^{(h)} I \right) + \sum_j \left(\int\limits_{k\in\mathbb{R}^d} \mathrm{D}k\, T_j^{(s)} I - \sum_i \int\limits_{k\in D_h} \mathrm{D}k\, T_i^{(h)} T_j^{(s)} I \right)$$

The expansions commute: $\boxed{T_i^{(h)} T_j^{(s)} I = T_j^{(s)} T_i^{(h)} I \equiv T_{i,j}^{(h,s)} I}$

$\Rightarrow$ **Identity:** $F = \underbrace{\sum_i \int \mathrm{D}k\, T_i^{(h)} I}_{F^{(h)}} + \underbrace{\sum_j \int \mathrm{D}k\, T_j^{(s)} I}_{F^{(s)}} - \underbrace{\sum_{i,j} \int \mathrm{D}k\, T_{i,j}^{(h,s)} I}_{F^{(h,s)}}$

All terms are integrated over the whole integration domain $\mathbb{R}^d$ as prescribed for the expansion by regions $\Rightarrow$ location of boundary Λ between D_h, D_s is irrelevant.

Identity: $$F = \underbrace{\sum_i \int \mathrm{D}k\, T_i^{(h)} I}_{F^{(h)}} + \underbrace{\sum_j \int \mathrm{D}k\, T_j^{(s)} I}_{F^{(s)}} - \underbrace{\sum_{i,j} \int \mathrm{D}k\, T_{i,j}^{(h,s)} I}_{F^{(h,s)}}$$



Additional overlap contribution $F^{(h,s)}$?

$$F^{(h,s)} = \sum_{i=0}^{\infty} (1+i) \sum_{j_1,j_2=0}^{\infty} (-1)^{j_2} \frac{(j_1+j_2)!}{j_1!\, j_2!} \frac{(m^2)^i}{(p^2)^{1+j_1+j_2}} \int \mathrm{D}k \frac{(-2k\cdot p)^{j_1}}{(k^2)^{2+i-j_2}} = 0 \quad \text{scaleless!}$$

Vanishing scaleless integrals ⇝ property of dimensional regularization and analytic continuation, not ad-hoc requirement of the formalism here!

Both UV- and IR-singularities are regularized dimensionally. Separate singularities:

$$F^{(h,s)} = \frac{1}{p^2}\left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}}\right) = 0$$

from $\int \frac{\mathrm{D}k}{(k^2)^2} = \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}}$ ⇝ cancels corresponding singularities in
$F^{(h)} = \frac{1}{p^2}\left(-\frac{1}{\epsilon_{\text{IR}}} + \mathcal{O}(\epsilon^0)\right)$ and $F^{(s)} = \frac{1}{p^2}\left(\frac{1}{\epsilon_{\text{UV}}} + \mathcal{O}(\epsilon^0)\right)$.
↪ Complete result $F = F^{(h)} + F^{(s)} - F^{(h,s)}$ is separately UV-finite and IR-finite.

$\Rightarrow$ $\boxed{F = F^{(h)} + F^{(s)}}$ as found before.

But now this identity has been obtained without evaluating F, $F^{(h)}$, $F^{(s)}$!

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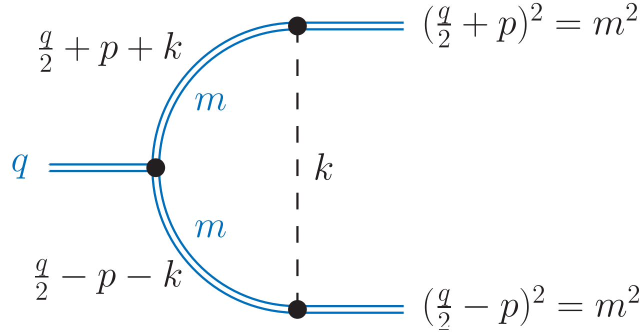
Example with 3 regions: threshold expansion for heavy-particle pair production

Regions analyzed in Beneke, Smirnov, NPB 522 (1998) 321



Centre-of-mass system: $(q^\mu)=(q_0,\vec{0})$, $(p^\mu)=(0,\vec{p})$

Close to threshold: $q^2\approx(2m)^2 \Rightarrow$ $q^2\gg|p^2|$ or $q_0\gg|\vec{p}|$

$$F=\int\frac{\mathrm{D}k}{(k^2+q_0k_0-2\vec{p}\cdot\vec{k})\,(k^2-q_0k_0-2\vec{p}\cdot\vec{k})\,k^2}$$

Relevant regions:

- **hard** (h): $k_0,|\vec{k}|\sim q_0 \Rightarrow$ expand $\sum_j T_j^{(h)}$ in $D_h=\Big\{k\in\mathbb{R}^d:\ |k_0|\gg|\vec{p}|$ or $|\vec{k}|\gg|\vec{p}|\Big\}$
- **soft** (s): $k_0,|\vec{k}|\sim|\vec{p}| \Rightarrow$ expand $\sum_j T_j^{(s)}$ in $D_s=\Big\{k\in\mathbb{R}^d:\ |\vec{k}|\lesssim|k_0|\lesssim|\vec{p}|\Big\}$
- **potential** (p): $k_0\sim\frac{\vec{p}^{\,2}}{q_0}$, $|\vec{k}|\sim|\vec{p}| \Rightarrow \sum_j T_j^{(p)}$ in $D_p=\Big\{k\in\mathbb{R}^d:\ |k_0|\ll|\vec{k}|\lesssim|\vec{p}|\Big\}$

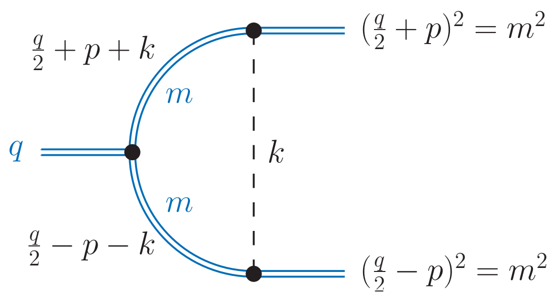
[no explicit boundaries needed]

$\hookrightarrow$ The expansion $T^{(x)}\equiv\sum_j T_j^{(x)}$ converges for $k\in D_x$ $(x=h,s,p)$.

$\hookrightarrow$ $D_h\cup D_s\cup D_p=\mathbb{R}^d$, $\quad D_h\cap D_s=D_h\cap D_p=D_s\cap D_p=\emptyset$

$\hookrightarrow$ The expansions $T^{(h)},T^{(s)},T^{(p)}$ commute with each other.

Threshold expansion (2)



Similar transformations as for the large-momentum example yield the following **identity:**

$$F = F^{(h)} + \underbrace{F^{(s)}}_{=0} + F^{(p)} - \Big(\underbrace{F^{(h,s)}}_{=0} + \underbrace{F^{(h,p)}}_{=0} + \underbrace{F^{(s,p)}}_{=0}\Big) + \underbrace{F^{(h,s,p)}}_{=0 \text{ (scaleless)}}$$

with

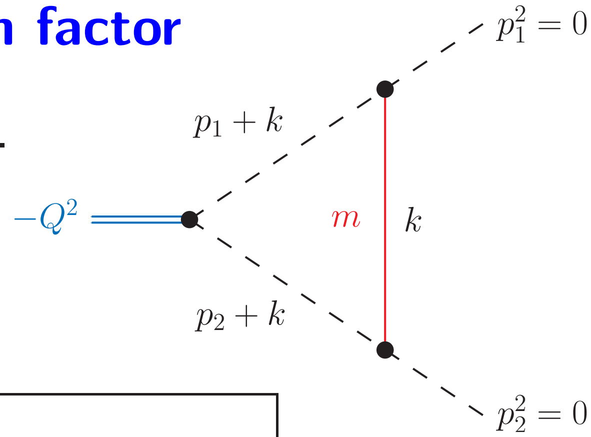
$$F^{(h)} = -\frac{2\, e^{\epsilon\gamma_E}\,\Gamma(\epsilon)}{q^2}\left(\frac{4\mu^2}{q^2}\right)^{\epsilon}\sum_{j=0}^{\infty}\left(-\frac{4p^2}{q^2}\right)^{j}\frac{(1+\epsilon)_j}{j!\,(1+2\epsilon+2j)}$$

$$F^{(p)} = \frac{e^{\epsilon\gamma_E}\,\Gamma(\frac{1}{2}+\epsilon)\,\sqrt{\pi}}{2\epsilon\,\sqrt{q^2\,(p^2-i0)}}\left(\frac{\mu^2}{p^2-i0}\right)^{\epsilon} \qquad \text{[higher orders vanish]}$$

Exact result reproduced:

$$F^{(h)} + F^{(p)} = F = \frac{e^{\epsilon\gamma_E}\,\Gamma(\epsilon)}{2p^2}\left(\frac{\mu^2}{p^2-i0}\right)^{\epsilon}{}_2F_1\left(\frac{1}{2}, 1+\epsilon; \frac{3}{2}; -\frac{q^2}{4p^2}-i0\right) \quad \checkmark$$

Example with non-commuting expansions: Sudakov form factor



Cannot always choose expansions which commute with each other.

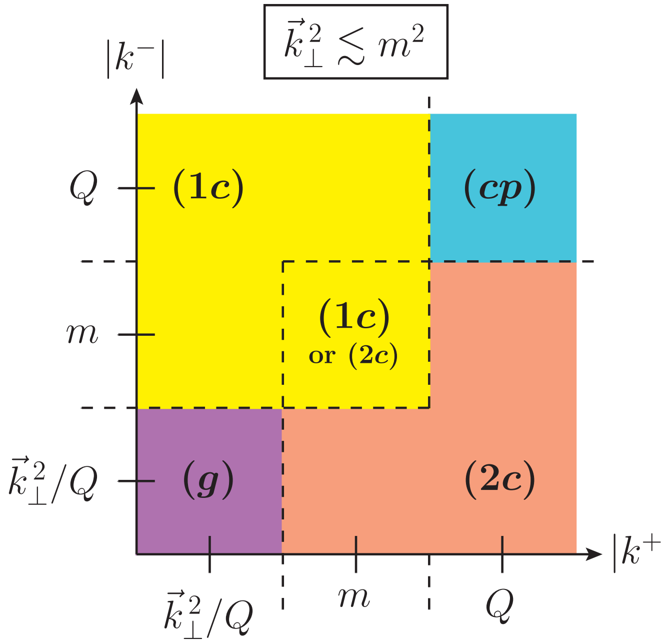
Sudakov limit: $-(p_1 - p_2)^2 = \boxed{Q^2 \gg m^2}$

$$F = \int \frac{\mathrm{D}k}{(k^+k^- - \vec{k}_\perp^2 + Qk^+)^{1+\delta}\,(k^+k^- - \vec{k}_\perp^2 + Qk^-)^{1-\delta}\,(k^+k^- - \vec{k}_\perp^2 - m^2)}$$

$\hookrightarrow$ analytic regulator $\delta \to 0$ [light-cone coordinates: $2p_{1,2} \cdot k = Qk^\pm$, $p_{1,2} \cdot k_\perp = 0$]

Regions & domains:

- **hard** (h): $k^+, k^-, |\vec{k}_\perp| \sim Q \Rightarrow D_h = \left\{k \in \mathbb{R}^d : \vec{k}_\perp^2 \gg m^2\right\}$
- **1-collinear** $(1c)$: $k^+ \sim \frac{m^2}{Q}$, $k^- \sim Q$, $|\vec{k}_\perp| \sim m$
- **2-collinear** $(2c)$: $k^+ \sim Q$, $k^- \sim \frac{m^2}{Q}$, $|\vec{k}_\perp| \sim m$
- **Glauber** (g): $k^+, k^- \sim \frac{m^2}{Q}$, $|\vec{k}_\perp| \sim m$
- **collinear-plane** (cp): $k^+, k^- \sim Q$, $|\vec{k}_\perp| \sim m$
  $\hookrightarrow$ "artificial" region to ensure $\cup_x D_x = \mathbb{R}^d$



[No soft region needed: $T^{(s)} \equiv T^{(1c)}T^{(2c)}$]

Most expansions commute, but $T^{(g)}T^{(cp)} \neq T^{(cp)}T^{(g)}$!

Sudakov form factor (2)

$T^{(g)}T^{(cp)} \neq T^{(cp)}T^{(g)} \rightsquigarrow$ Construct **identity** avoiding combination of (g) and (cp):

$$
\begin{aligned}
F = {} & F^{(h)} + F^{(1c)} + F^{(2c)} + F^{(g)} + F^{(cp)} \\
& - \Big(F^{(h,1c)} + F^{(h,2c)} + F^{(h,g)} + F^{(h,cp)} + F^{(1c,2c)} + F^{(1c,g)} + F^{(1c,cp)} + F^{(2c,g)} + F^{(2c,cp)} \Big) \\
& + F^{(h,1c,2c)} + F^{(h,1c,g)} + F^{(h,1c,cp)} + F^{(h,2c,g)} + F^{(h,2c,cp)} + F^{(1c,2c,g)} + F^{(1c,2c,cp)} \\
& - \Big(F^{(h,1c,2c,g)} + F^{(h,1c,2c,cp)} \Big) + F^{\text{extra}}_{cp\leftarrow g} + F^{\text{extra}}_{g\leftarrow cp}
\end{aligned}
$$

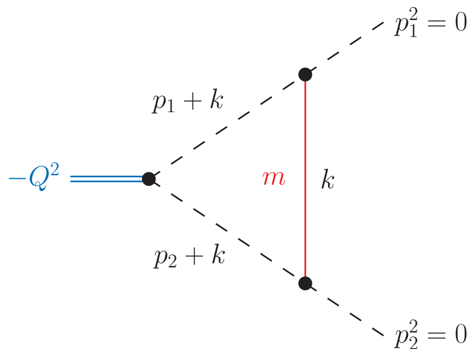
Usual terms:

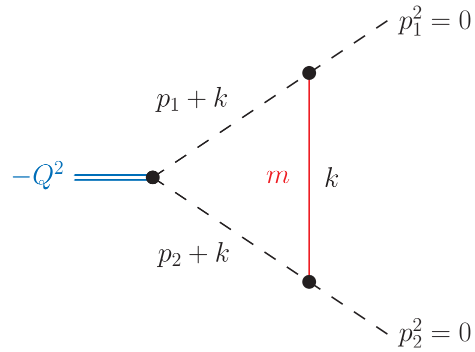
- no combination of (g) and (cp)
- $F^{(g)}$, $F^{(cp)}$ and all overlap contributions are scaleless (with analytic regularization)

Extra terms:

- $F^{\text{extra}}_{cp\leftarrow g}$ involves $T^{(cp)}T^{(g)}$ integrated over $k \in D_{cp}$,
- $F^{\text{extra}}_{g\leftarrow cp}$ involves $T^{(g)}T^{(cp)}$ integrated over $k \in D_g$,

plus all combinations of $T^{(h)}, T^{(1c)}, T^{(2c)}$, with alternating signs.





Sudakov form factor (3)

Both extra terms cancel at the integrand level:

$$F^{\text{extra}}_{g\leftarrow cp} = \int_{k\in D_g} \mathrm{D}k \Big(-1 + T^{(h)} + T^{(1c)} + T^{(2c)} - T^{(h,1c)} - T^{(h,2c)} - T^{(1c,2c)} + T^{(h,1c,2c)}\Big) T^{(g)} T^{(cp)} I$$

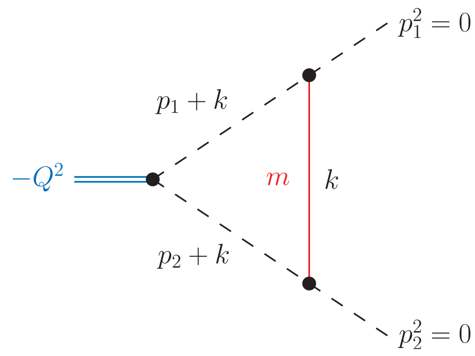
$$= (-1+3-3+1) \int_{k\in D_g} \mathrm{D}k\, T^{(g)} T^{(cp)} I = 0$$

because $T^{(x)} T^{(g)} T^{(cp)} = T^{(g)} T^{(cp)}$ $\forall x \in \{h, 1c, 2c\}$.

Similarly: $F^{\text{extra}}_{cp\leftarrow g} = 0$.

[The extra terms must cancel $\rightsquigarrow$ otherwise dependence on boundaries of D_g, D_{cp}.]

Sudakov form factor (4)



Omitting scaleless contributions and vanishing extra terms:

$$\boxed{F = F^{(h)} + F^{(1c)} + F^{(2c)}}$$

Regions explicitly evaluated to all orders in $\frac{m^2}{Q^2}$: [omitting $\mathcal{O}(\delta)$ and $\mathcal{O}(\epsilon)$]

$$F^{(h)} = -\frac{1}{Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\left\{\frac{1}{\epsilon^2} - \frac{2}{\epsilon}\ln\left(1-\frac{m^2}{Q^2}\right) + \ln^2\left(1-\frac{m^2}{Q^2}\right) - 2\,\mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) - \frac{\pi^2}{12}\right\}$$

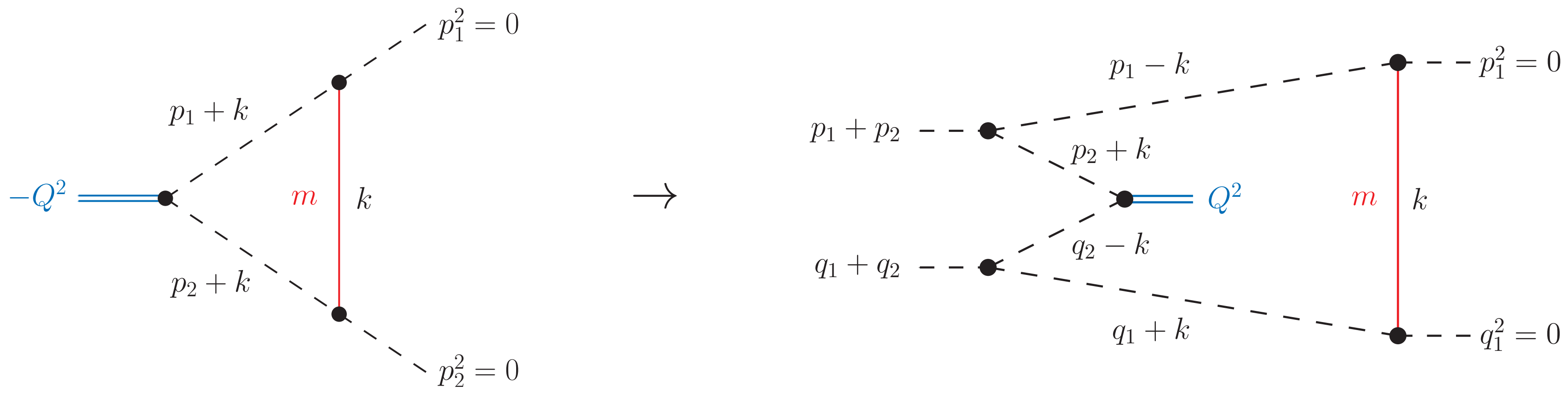
$$F^{(1c)}, F^{(2c)} = -\frac{1}{2Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\left\{\pm\frac{1}{\delta}\left[\frac{1}{\epsilon} + \ln\frac{Q^2}{m^2} - \ln\left(1-\frac{m^2}{Q^2}\right)\right] - \frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\left(1-\frac{m^2}{Q^2}\right)\right.$$
$$\left. + \frac{1}{2}\ln^2\frac{Q^2}{m^2} + \ln\frac{Q^2}{m^2}\ln\left(1-\frac{m^2}{Q^2}\right) - \ln^2\left(1-\frac{m^2}{Q^2}\right) + \mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) + \frac{5}{12}\pi^2\right\}$$

$\hookrightarrow$ $F^{(1c)}$ and $F^{(2c)}$ are not separately finite for $\delta \to 0$, but their sum is.

Compare to exact result:

$$F = -\frac{1}{Q^2}\left\{\frac{1}{2}\ln^2\frac{Q^2}{m^2} + \ln\frac{Q^2}{m^2}\ln\left(1-\frac{m^2}{Q^2}\right) - \mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) + \frac{\pi^2}{3}\right\} \quad \checkmark$$

Sudakov form factor → 5-point integral with Glauber contribution



- collinear propagators “doubled”, but expansions equivalent
- same regions & domains
- “double” propagators ⇝ Glauber contribution present (even with analytic regularization)
- leading contributions:

$$F_0^{(g)} \propto \frac{1}{(Q^2)^3} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left(\frac{m^2}{Q^2}\right)^{-2-\epsilon}$$

$$F_0^{(1c)}, F_0^{(2c)} \propto \frac{1}{(Q^2)^3} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left(\frac{m^2}{Q^2}\right)^{-1-\epsilon}$$

$$F_0^{(h)} \propto \frac{1}{(Q^2)^3} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon}$$

Overview

I The strategy of regions

- asymptotic expansion of loop integrals
- introduction to the expansion by regions
- example: large-momentum expansion

II Why does the method work?

- transforming original integral $\rightarrow$ series of expanded integrals
- overlap contribution

III Examples

- threshold expansion for heavy-particle pair production
- Sudakov form factor ($\rightsquigarrow$ non-commuting expansions)

IV The general formalism

- conditions on regions & expansions
- general identity with overlap contributions
- example: forward scattering with small momentum exchange

V Summary

IV The general formalism

Identities as in the previous examples are generally valid, under some conditions.

Consider

- a (multiple) integral $F = \int \mathrm{D}k\, I$ over the domain D (e.g. $D = \mathbb{R}^d$),
- a set of N regions $R = \{x_1, \ldots, x_N\}$,
- for each region $x \in R$ an expansion $T^{(x)} = \sum_j T_j^{(x)}$ which converges absolutely in the domain $D_x \subset D$.

Conditions

- $\bigcup_{x \in R} D_x = D, \qquad D_x \cap D_{x'} = \emptyset \;\forall x \neq x'\,.$
- Some of the expansions commute with each other.
  Let $R_\mathrm{c} = \{x_1, \ldots, x_{N_\mathrm{c}}\}$ and $R_\mathrm{nc} = \{x_{N_\mathrm{c}+1}, \ldots, x_N\}$ with $1 \leq N_\mathrm{c} \leq N$.
  Then: $T^{(x)} T^{(x')} = T^{(x')} T^{(x)} \equiv T^{(x,x')} \;\forall x \in R_\mathrm{c}\,,\; x' \in R\,.$
- Every pair of non-commuting expansions is invariant under some expansion from R_c:
  $\forall x'_1, x'_2 \in R_\mathrm{nc},\; x'_1 \neq x'_2,\; \exists x \in R_\mathrm{c} : T^{(x)} T^{(x'_2)} T^{(x'_1)} = T^{(x'_2)} T^{(x'_1)}\,.$
- $\exists$ regularization for singularities, e.g. dimensional (+ analytic) regularization.
  $\hookrightarrow$ All expanded integrals and series expansions in the formalism are well-defined.

The general formalism (2)

Under these conditions, the following **identity** holds: $\left[F^{(x,\dots)} \equiv \sum_{j,\dots} \int \mathrm{D}k\, T^{(x,\dots)}_{j,\dots} I\right]$

$$F = \sum_{x\,\in\,R} F^{(x)} - \sum_{\{x_1',\,x_2'\}\,\subset\,R}^{\langle R_\mathsf{c}+1\rangle} F^{(x_1',x_2')} + \ldots - (-1)^n \sum_{\{x_1',\,\ldots,\,x_n'\}\,\subset\,R}^{\langle R_\mathsf{c}+1\rangle} F^{(x_1',\ldots,x_n')} + \ldots + (-1)^{N_\mathsf{c}} \sum_{x'\,\in\,R_\mathsf{nc}} F^{(x',x_1,\ldots,x_{N_\mathsf{c}})}$$

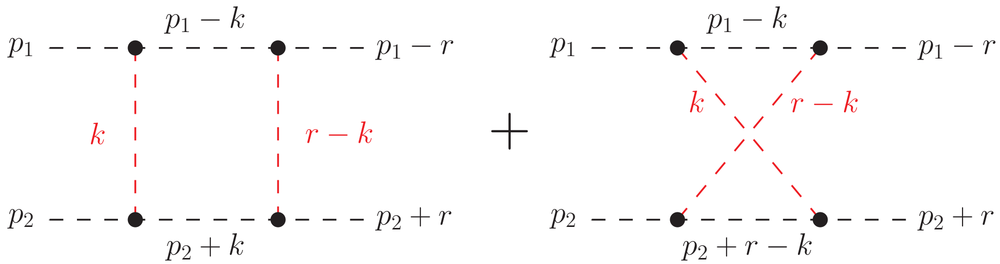
where the sums run over subsets $\{x_1', \ldots\}$ containing at most one region from R_nc.

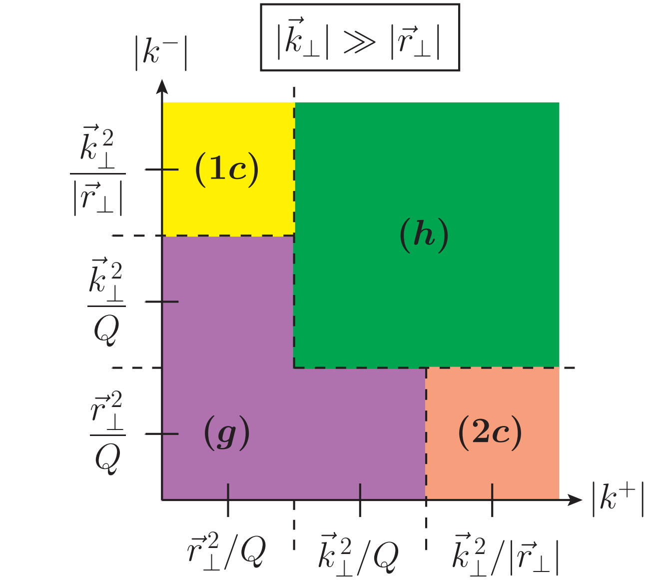
Comments

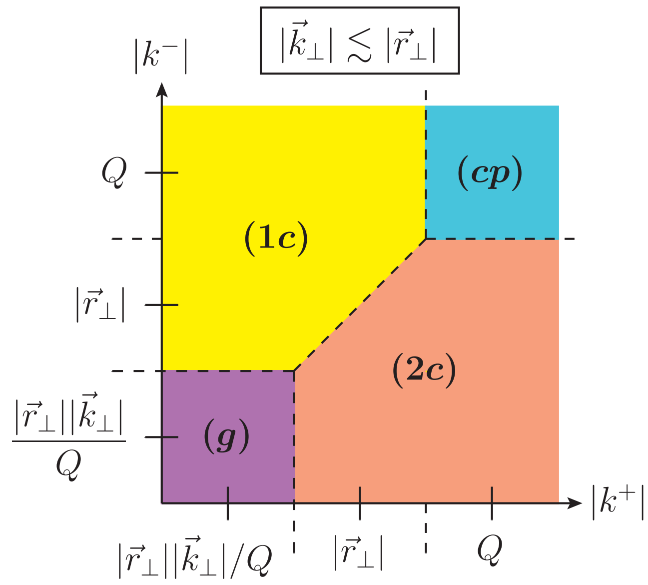
- This identity is exact when the expansions are summed to all orders. ✓
  Leading-order approximation for $F \rightsquigarrow$ dropping higher-order terms.
- It is independent of the regularization (dim. reg., analytic reg., cut-off, infinitesimal masses/off-shellness, ...) as long as all individual terms are well-defined.
- Usually regions & regularization are chosen such that multiple expansions $F^{(x_1',\ldots,x_n')}$ $(n \geq 2)$ are scaleless and vanish.
  [✓ if each $F_0^{(x)}$ is a *homogeneous* function of the expansion parameter with *unique scaling*.]
- If $\exists\, F^{(x_1',x_2',\ldots)} \neq 0 \rightsquigarrow$ relevant overlap contributions ($\rightarrow$ "zero-bin subtractions").
  They appear e.g. when avoiding analytic regularization in SCET.

e.g. Manohar, Stewart '06;
Chiu, Fuhrer, Hoang, Kelley, Manohar '09; ...

Example with relevant overlap contributions: forward scattering with small momentum exchange







Two light-like particles with large centre-of-mass energy exchange a small momentum r:

$$p_1^2 = (p_1 - r)^2 = p_2^2 = (p_2 + r)^2 = 0$$

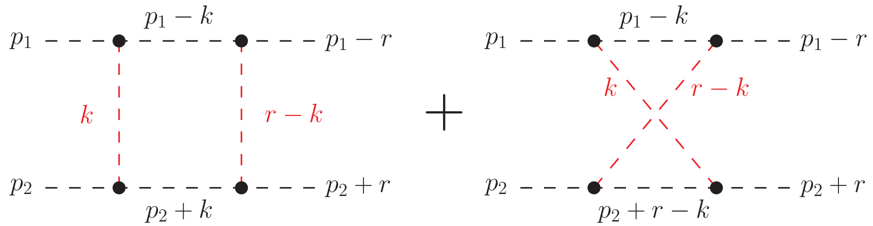
$$(p_1 + p_2)^2 = \boxed{Q^2 \gg \vec{r}_\perp^{\,2}}, \qquad r^\pm \approx \mp \frac{\vec{r}_\perp^{\,2}}{Q}$$

Symmetrize integral under $k \leftrightarrow r - k$
$\hookrightarrow$ avoids divergences at $|k^\pm| \to \infty$ under expansion.

$$\boxed{\begin{aligned} F = \frac{1}{2} \int \frac{\mathrm{D}k}{k^2\,(r-k)^2} \left(\frac{1}{\big((p_1 - k)^2\big)^{1+\delta}} + \frac{1}{\big((p_1 - r + k)^2\big)^{1+\delta}} \right) \\ \times \left(\frac{1}{\big((p_2 + k)^2\big)^{1-\delta}} + \frac{1}{\big((p_2 + r - k)^2\big)^{1-\delta}} \right) \end{aligned}}$$

Regions: same as for Sudakov form factor (scaling with $m \to |\vec{r}_\perp|$),
Domains: similar (but more involved for $|\vec{k}_\perp| \gg |\vec{r}_\perp|$)

Forward scattering (2)



Same identity as for Sudakov form factor:

$$F = F^{(h)} + F^{(1c)} + F^{(2c)} + F^{(g)} + F^{(cp)} - \left(F^{(h,1c)} + F^{(h,2c)} + F^{(h,g)} + F^{(h,cp)} + F^{(1c,2c)} + F^{(1c,g)} + F^{(1c,cp)} + F^{(2c,g)} + F^{(2c,cp)}\right) + F^{(h,1c,2c)} + F^{(h,1c,g)} + F^{(h,1c,cp)} + F^{(h,2c,g)} + F^{(h,2c,cp)} + F^{(1c,2c,g)} + F^{(1c,2c,cp)} - \left(F^{(h,1c,2c,g)} + F^{(h,1c,2c,cp)}\right)$$

With analytic regulator $\delta \to 0$: $\boxed{F_0 = F_0^{(1c)} + F_0^{(2c)}}$ [$F_0^{(h)}$ suppressed, others scaleless]

$$F_0^{(1c)} = F_0^{(2c)} = \frac{1}{2}\,\frac{i\pi}{Q^2\,\vec{r}_\perp^{\,2}}\left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon}\frac{e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}$$

Without analytic regularization ($\delta = 0$): [all terms are still well-defined]

$$\boxed{F_0 = F_0^{(1c)} + F_0^{(2c)} + F_0^{(g)} - \left(F_0^{(1c,2c)} + F_0^{(1c,g)} + F_0^{(2c,g)}\right) + F_0^{(1c,2c,g)}}$$

$$F_0^{(x,\ldots)} = \frac{i\pi}{Q^2\,\vec{r}_\perp^{\,2}}\left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon}\frac{e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} \qquad \forall\,\{x,\ldots\}\subset\{1c,2c,g\}$$

↪ consistent results independent of regularization: $\frac{1}{2} + \frac{1}{2} = 1 + 1 + 1 - (1+1+1) + 1$ ✓

↪ agreement with leading-order expansion of full result

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Expansion by regions for general integrals

- Conditions for regions (+ corresponding expansions & domains) established.
- Identity proven ⇝ relates exact integral to sum of expanded terms:

$$F = \sum_{x \in R} F^{(x)} - \sum_{\{x'_1, x'_2\} \subset R}^{\langle R_c + 1\rangle} F^{(x'_1, x'_2)} + \ldots - (-1)^n \sum_{\{x'_1, \ldots, x'_n\} \subset R}^{\langle R_c + 1\rangle} F^{(x'_1, \ldots, x'_n)} + \ldots + (-1)^{N_c} \sum_{x' \in R_{nc}} F^{(x', x_1, \ldots, x_{N_c})}$$

↪ valid independent of the choice of regularization

- This identity includes overlap contributions with multiple expansions

↪ can be scaleless ⇝ known recipe for expansion by regions ✓

or relevant (depending on regularization)

↪ generalization of known recipe.

Application to example integrals

- setup of the regions, expansions & convergence domains,
- check of conditions,
- evaluation of expanded integrals,
- comparison to exact result.