High Precision for Hard Processes (HP^2)
Munich, September 4–7, 2012

Expansion by regions: foundation, generalization and automated search for regions

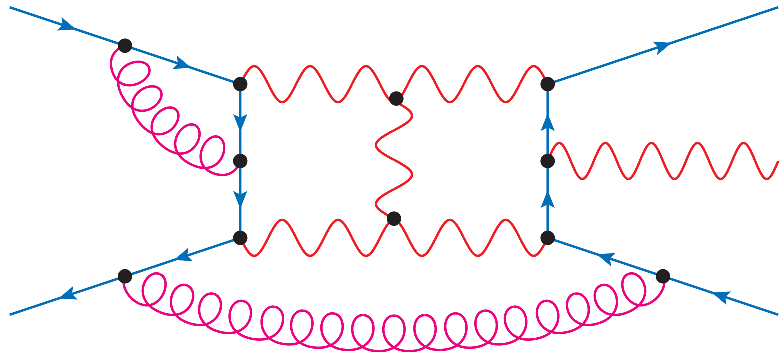
Bernd Jantzen
RWTH Aachen University

Based on:

II–III B.J., JHEP 12 (2011) 076, arXiv:1111.2589

IV B.J., A. Smirnov, V. Smirnov, arXiv:1206.0546 $\rightsquigarrow$ Eur. Phys. J. C 72 (2012) 2139

I The strategy of expansion by regions



Starting point: (multi-)loop integral

(or other complicated integral)

$$F = \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \cdots I \,,$$

$$I = \frac{1}{(k_1+p_1)^2 - m_1^2} \, \frac{1}{(k_1+k_2+p_2)^2 - m_2^2} \cdots$$

- complicated function of internal masses m_i and kinematical parameters p_i^2, $p_i \cdot p_j$
- exact evaluation often hard or impossible

Exploit parameter hierarchies, e.g. large energies $Q \gg$ small masses m:

- expand integral in small ratios $\frac{m^2}{Q^2}$: $F = F_0 + \frac{m^2}{Q^2} F_1 + \left(\frac{m^2}{Q^2}\right)^2 F_2 + \ldots$
- simplification achieved if expansion of integrand before integration:
  $I \to I_0 + \frac{m^2}{Q^2} I_1 + \left(\frac{m^2}{Q^2}\right)^2 I_2 + \ldots \,,$ $\qquad F_j = \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \cdots I_j$
- expanded integrands I_j often simpler to integrate than original integrand I

Expansion of integrand before integration?

$$I \to I_0 + \frac{m^2}{Q^2} I_1 + \left(\frac{m^2}{Q^2}\right)^2 I_2 + \ldots \,, \qquad F_j = \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \cdots I_j$$

But:

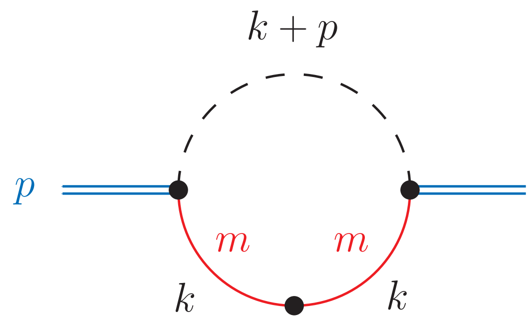
- ⋆ integrand I is function of loop momenta: $I = I(k_1, k_2, \ldots)$
- ⋆ loop-momentum components k_i^μ can take any values (large, small, mixed, ...)
- ⋆ expansions of integrand may break down for certain values of $k_1, k_2, \ldots$
- ⋆ naive integrations of expanded integrand may generate new singularities

↪ Need sophisticated methods of asymptotic expansions.

Simple example: large-momentum expansion

$$F = \int \frac{\mathrm{D}k}{(k+p)^2\,(k^2-m^2)^2}$$

$$\left[\int \mathrm{D}k \equiv \frac{\mu^{2\epsilon} e^{\epsilon\gamma_E}}{i\pi^{d/2}} \int \mathrm{d}^d k \qquad d = 4-2\epsilon\right]$$



Large momentum $|p^2| \gg m^2$ $\rightsquigarrow$ expand in $\frac{m^2}{p^2}$.

Integral is UV- and IR-finite, the exact result is known: $[p^2 \to p^2 + i0]$

$$F = \frac{1}{p^2}\left[\ln\left(\frac{-p^2}{m^2}\right) + \ln\left(1 - \frac{m^2}{p^2}\right)\right] + \mathcal{O}(\epsilon)$$

$$\xrightarrow[\text{expand}]{} \frac{1}{p^2}\left[\ln\left(\frac{-p^2}{m^2}\right) - \sum_{j=1}^{\infty} \frac{1}{j}\left(\frac{m^2}{p^2}\right)^j\right] + \mathcal{O}(\epsilon)$$

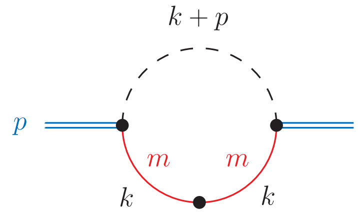
Now assume that we could <u>not</u> calculate this integral exactly . . .

Large-momentum expansion (2)

Large momentum $|p^2| \gg m^2$

$\hookrightarrow$ expand integrand before integration:

$$F = \int \frac{\mathrm{D}k}{(k+p)^2\,(k^2-m^2)^2}$$



Expansion by regions

Beneke, V. Smirnov, Nucl. Phys. B 522 (1998) 321
V. Smirnov, Rakhmetov, Theor. Math. Phys. 120 (1999) 870
V. Smirnov, Phys. Lett. B 465 (1999) 226

$\hookrightarrow$ here 2 relevant **regions**:

- **hard** (h): $k \sim p \Rightarrow \dfrac{1}{(k+p)^2\,(k^2-m^2)^2} \to \dfrac{1}{(k+p)^2}\left(\dfrac{1}{(k^2)^2} + \dfrac{2m^2}{(k^2)^3} + \ldots\right)$
- **soft** (s): $k \sim m \Rightarrow \dfrac{1}{(k+p)^2\,(k^2-m^2)^2} \to \dfrac{1}{(k^2-m^2)^2}\left(\dfrac{1}{p^2} - \dfrac{2k\cdot p}{(p^2)^2} - \dfrac{k^2}{(p^2)^2} + \ldots\right)$

$\Rightarrow$ Integrate each expanded term over the **whole integration domain**.

$\Rightarrow$ Set scaleless integrals to zero (like in dimensional regularization).

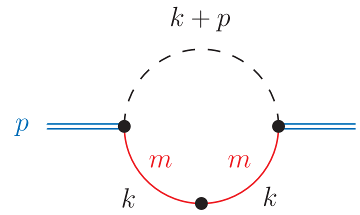
Leading-order contributions:

- **hard:** $F_0^{(h)} = \displaystyle\int \frac{\mathrm{D}k}{(k+p)^2\,(k^2)^2} = \frac{1}{p^2}\left(\frac{\mu^2}{-p^2}\right)^{\epsilon}\left(-\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right)$
- **soft:** $F_0^{(s)} = \displaystyle\int \frac{\mathrm{D}k}{p^2\,(k^2-m^2)^2} = \frac{1}{p^2}\left(\frac{\mu^2}{-p^2}\right)^{\epsilon}\left(\frac{m^2}{-p^2}\right)^{-\epsilon}\left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right)$

$\hookrightarrow$ Contributions are homogeneous functions of the expansion parameter $\frac{m^2}{p^2}$.

Large-momentum expansion (3)

$$F = \int \frac{\mathrm{D}k}{(k+p)^2\,(k^2-m^2)^2}$$



Leading-order contributions:

- **hard:** $F_0^{(h)} = \int \frac{\mathrm{D}k}{(k+p)^2\,(k^2)^2} = \frac{1}{p^2}\left[-\frac{1}{\epsilon} + \ln\left(\frac{-p^2}{\mu^2}\right)\right] + \mathcal{O}(\epsilon) \rightsquigarrow$ IR-singular!
- **soft:** $F_0^{(s)} = \frac{1}{p^2}\int \frac{\mathrm{D}k}{(k^2-m^2)^2} = \frac{1}{p^2}\left[\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right)\right] + \mathcal{O}(\epsilon) \rightsquigarrow$ UV-singular!

↪ Singularities are cancelled in the sum of all contributions.

↪ Exact result is approximated:

$$F_0 = F_0^{(h)} + F_0^{(s)} = \frac{1}{p^2}\ln\left(\frac{-p^2}{m^2}\right) + \mathcal{O}(\epsilon) = F + \mathcal{O}\left(\frac{m^2}{(p^2)^2}\right) \quad \checkmark$$

Hard & soft expansions to all orders in $\frac{m^2}{p^2}$ $\rightsquigarrow$ exact result F reproduced ✓

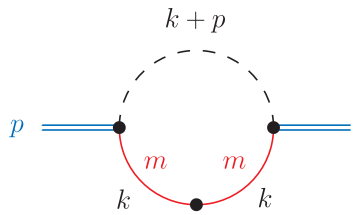
Expansion by regions: successfully applied to many complicated loop integrals

But: Why does it work?

- What ensures the cancellation of singularities? (IR $\leftrightarrow$ UV!)
- Didn't we double-count every part of the integration domain when replacing $\int \mathrm{D}k\, I \to \int \mathrm{D}k\, I_0^{(h)} + \int \mathrm{D}k\, I_0^{(s)}$?
- How do we have to choose the regions?
  And how do we know that the chosen set of regions is complete?
- What is the role of scaleless integrals?

II Why does the method work?

Idea based on a 1-dimensional toy example from M. Beneke (1997)
[see also: V. Smirnov, *Applied Asymptotic Expansions In Momenta And Masses*]



Large-momentum example

Let us show step by step how the expansions reproduce the full result.

The hard & soft expansions converge absolutely within domains D_h, D_s:

$$(h):\ \frac{1}{(k^2-m^2)^2} = \sum_i T_i^{(h)} \frac{1}{(k^2-m^2)^2} \text{ within } D_h = \left\{ k \in \mathbb{R}^d : |k^2| \geq \Lambda^2 \right\},$$

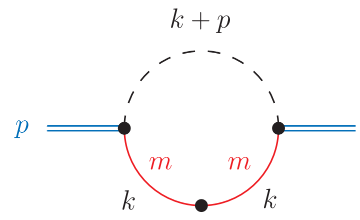
$$(s):\ \frac{1}{(k+p)^2} = \sum_j T_j^{(s)} \frac{1}{(k+p)^2} \text{ within } D_s = \left\{ k \in \mathbb{R}^d : |k^2| < \Lambda^2 \right\},$$

with $m^2 \ll \Lambda^2 \ll |p^2| \rightsquigarrow D_h \cup D_s = \mathbb{R}^d \quad [D_h \cap D_s = \emptyset]$.

The expansions commute with integrals restricted to the corresponding domains:

$$\int\limits_{k\in D_h} \mathrm{D}k \underbrace{\frac{1}{(k+p)^2\,(k^2-m^2)^2}}_{I} = \sum_i \int\limits_{k\in D_h} \mathrm{D}k\, T_i^{(h)} I\,, \qquad \int\limits_{k\in D_s} \mathrm{D}k\, I = \sum_j \int\limits_{k\in D_s} \mathrm{D}k\, T_j^{(s)} I$$

Transform the expression for the full integral:



$$F = \int\limits_{k\in D_h} \mathrm{D}k\, I + \int\limits_{k\in D_s} \mathrm{D}k\, I = \sum_i \int\limits_{k\in D_h} \mathrm{D}k\, T_i^{(h)} I + \sum_j \int\limits_{k\in D_s} \mathrm{D}k\, T_j^{(s)} I$$

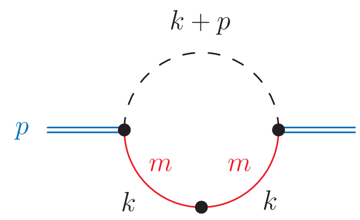
$$= \sum_i \Bigg(\int\limits_{k\in\mathbb{R}^d} \mathrm{D}k\, T_i^{(h)} I - \sum_j \int\limits_{k\in D_s} \mathrm{D}k\, T_j^{(s)} T_i^{(h)} I \Bigg) + \sum_j \Bigg(\int\limits_{k\in\mathbb{R}^d} \mathrm{D}k\, T_j^{(s)} I - \sum_i \int\limits_{k\in D_h} \mathrm{D}k\, T_i^{(h)} T_j^{(s)} I \Bigg)$$

The expansions commute: $\boxed{T_i^{(h)} T_j^{(s)} I = T_j^{(s)} T_i^{(h)} I \equiv T_{i,j}^{(h,s)} I}$

$\Rightarrow$ **Identity:** $F = \underbrace{\sum_i \int \mathrm{D}k\, T_i^{(h)} I}_{F^{(h)}} + \underbrace{\sum_j \int \mathrm{D}k\, T_j^{(s)} I}_{F^{(s)}} - \underbrace{\sum_{i,j} \int \mathrm{D}k\, T_{i,j}^{(h,s)} I}_{F^{(h,s)}}$

All terms are integrated over the whole integration domain $\mathbb{R}^d$ as prescribed for the expansion by regions $\Rightarrow$ location of boundary Λ between D_h, D_s is irrelevant.

Identity:
$$F = \underbrace{\sum_i \int \mathrm{D}k\, T_i^{(h)} I}_{F^{(h)}} + \underbrace{\sum_j \int \mathrm{D}k\, T_j^{(s)} I}_{F^{(s)}} - \underbrace{\sum_{i,j} \int \mathrm{D}k\, T_{i,j}^{(h,s)} I}_{F^{(h,s)}}$$



Additional overlap contribution $F^{(h,s)}$?

$$F^{(h,s)} = \sum_{i=0}^{\infty} (1+i) \sum_{j_1,j_2=0}^{\infty} (-1)^{j_2} \frac{(j_1+j_2)!}{j_1!\,j_2!} \frac{(m^2)^i}{(p^2)^{1+j_1+j_2}} \int \mathrm{D}k \frac{(-2k\cdot p)^{j_1}}{(k^2)^{2+i-j_2}} = 0 \quad \text{scaleless!}$$

Vanishing scaleless integrals ⇝ property of dimensional regularization and analytic continuation, not ad-hoc requirement of the formalism here!

Both UV- and IR-singularities are regularized dimensionally. Separate singularities:

$$F^{(h,s)} = \frac{1}{p^2}\left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}}\right) = 0$$

⇝ cancels corresponding singularities in $F^{(h)} = \frac{1}{p^2}\Big(-\frac{1}{\epsilon_{\text{IR}}} + \mathcal{O}(\epsilon^0)\Big)$ and $F^{(s)} = \frac{1}{p^2}\Big(\frac{1}{\epsilon_{\text{UV}}} + \mathcal{O}(\epsilon^0)\Big)$.

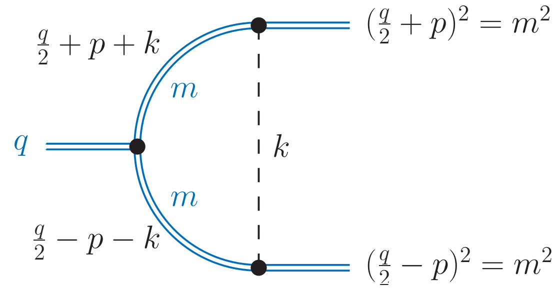
↪ Complete result $F = F^{(h)} + F^{(s)} - F^{(h,s)}$ is separately UV-finite and IR-finite.

$\Rightarrow$ $\boxed{F = F^{(h)} + F^{(s)}}$ as used before.

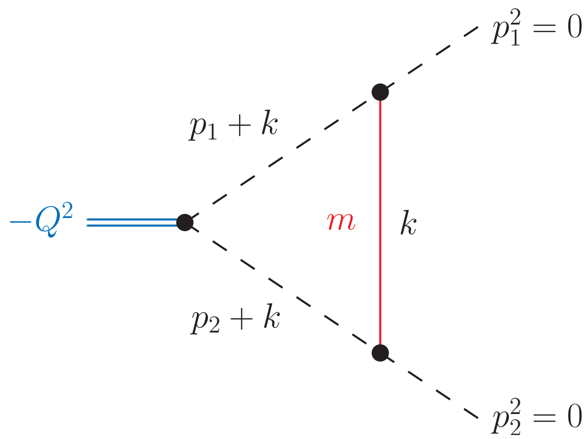
But now this identity has been obtained without evaluating the contributions!

More 1-loop examples

similar transformations applied ⇝ similar identities obtained

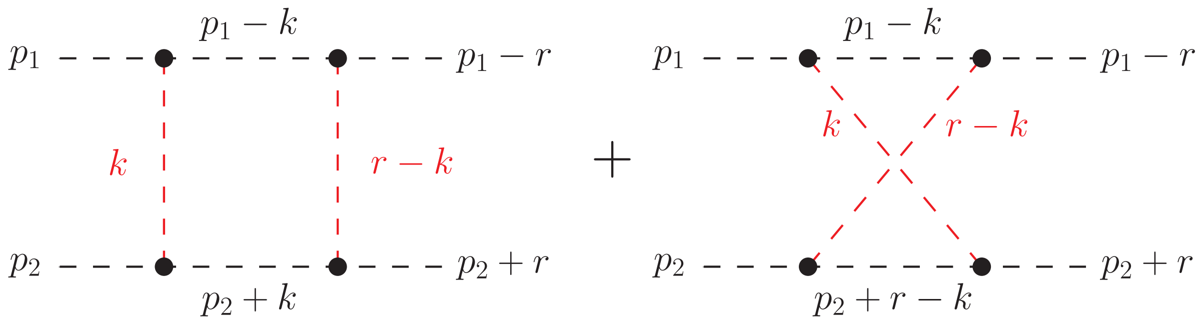


- Threshold expansion for heavy-particle pair production
  ↪ 3 regions with commuting expansions



- Sudakov form factor
  ↪ 5 regions, 2 non-commuting expansions

- Forward scattering with small momentum exchange
  ↪ overlap contributions eventually relevant



Non-commuting expansions: $T^{(x_1)}T^{(x_2)} \neq T^{(x_2)}T^{(x_1)}$

What changes if (some) expansions do not commute with each other?

$\hookrightarrow$ **identity** with combinations only of commuting expansions.

$\hookrightarrow$ **extra terms** involving pairs of non-commuting expansions, e.g.

$$-\int\limits_{k\in D_{x_2}} \mathrm{D}k \left(T^{(x_2)}T^{(x_1)} - T^{(x)}T^{(x_2)}T^{(x_1)} + \ldots \right) I$$

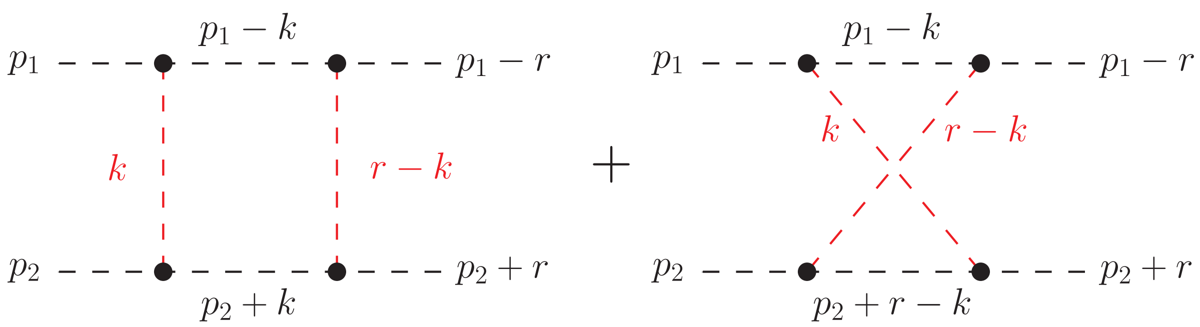
$\Rightarrow$ extra terms cancel at integrand level if

$\exists$ commuting expansion $T^{(x)}$ such that $\boxed{T^{(x)}T^{(x_2)}T^{(x_1)} = T^{(x_2)}T^{(x_1)}}$

This condition can usually be fulfilled. ✓

$\hookrightarrow$ **no extra terms**!

Example with relevant overlap contributions: forward scattering with small momentum exchange



$\hookrightarrow$ General identity with 5 regions + overlap contributions.

$\hookrightarrow$ Evaluation of terms depends on regularization scheme:
[restricting to leading order F_0]

- **Without analytic regularization:**

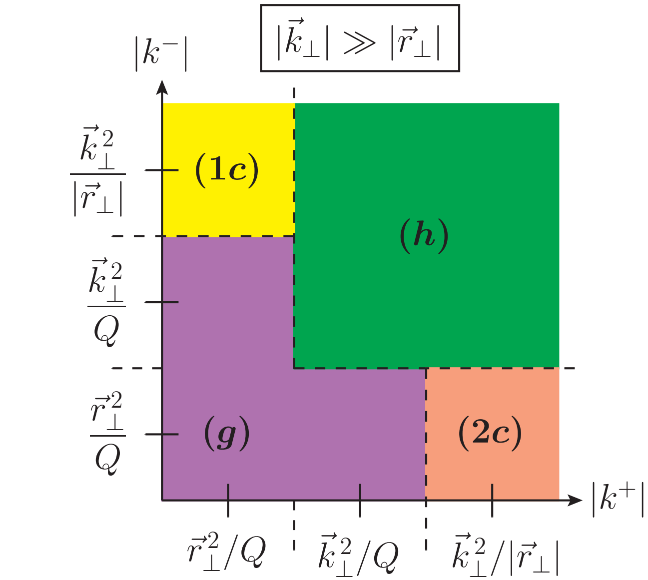
single expansions

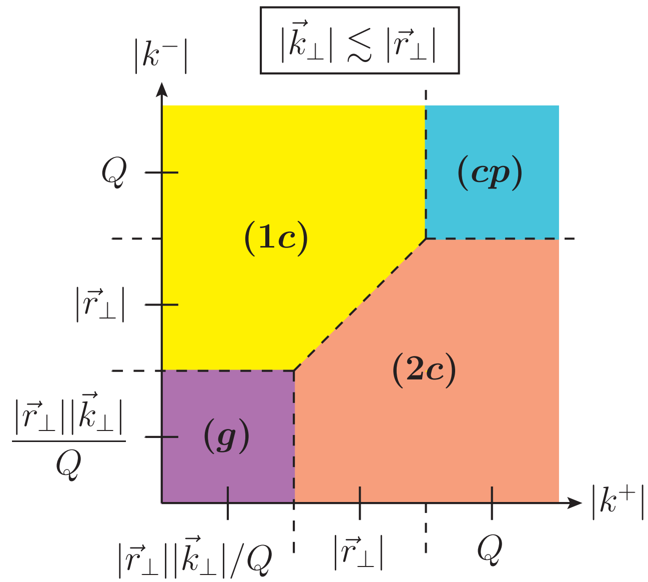
$$F_0 = F_0^{(1c)} + F_0^{(2c)} + F_0^{(g)} - \left(F_0^{(1c,2c)} + F_0^{(1c,g)} + F_0^{(2c,g)}\right) + F_0^{(1c,2c,g)}$$

relevant overlap contributions

- **With analytic regularization:** $F_0 = F_0^{(1c)} + F_0^{(2c)}$
  other terms scaleless

$\hookrightarrow$ Individual terms differ, but complete result agrees. ✓





III The general formalism

Consider

- a (multiple) integral F over the domain D,
- a set of regions $x_1, \ldots, x_N$,
- for each region x an expansion $T^{(x)}$ converging in the subdomain D_x.

Conditions

- The convergence domains D_x cover the integration domain D.
- If some expansions do not commute with each other:
  Every pair $T^{(x_2)}, T^{(x_1)}$ of non-commuting expansions is invariant under a commuting expansion $T^{(x)}$: $T^{(x)} T^{(x_2)} T^{(x_1)} = T^{(x_2)} T^{(x_1)}$
- All expanded integrals and series expansions are well-defined $\rightsquigarrow$ $\exists$ regularization.

$\hookrightarrow$ The following **identity** holds:

$$F = \left\{ \begin{array}{c} \text{single} \\ \text{expansions} \end{array} \right\} - \left\{ \begin{array}{c} \text{double} \\ \text{expansions} \end{array} \right\} + \left\{ \begin{array}{c} \text{triple} \\ \text{expansions} \end{array} \right\} - \ldots$$

where only those expansions are combined which commute with each other.

The general formalism (2)

$$F = \left\{\begin{matrix}\text{single}\\ \text{expansions}\end{matrix}\right\} \underbrace{- \left\{\begin{matrix}\text{double}\\ \text{expansions}\end{matrix}\right\} + \left\{\begin{matrix}\text{triple}\\ \text{expansions}\end{matrix}\right\} - \ldots}_{\text{overlap contributions}}$$

with those combinations of expansions which commute with each other

Comments

- Identity is exact when expansions are summed to all orders. ✓
  Want leading-order approximation? $\rightsquigarrow$ drop higher-order terms.
- Identity is independent of regularization.
  $\hookrightarrow$ Individual terms change with regularization, but complete result invariant.
- Overlap contributions ($\to$ "zero-bin subtractions") may be relevant.
  [e.g. when avoiding analytic regularization in SCET] e.g. Manohar, Stewart '06; Chiu, Fuhrer, Hoang, Kelley, Manohar '09; . . .
- With usual choice of regions & regularization
  $\hookrightarrow$ overlap contributions are scaleless and vanish.
  [✓ if single expansions yield *homogeneous* functions of expansion parameter with *unique scalings*.]

IV Automated search for regions with `asy2.m`

Now we have a proof for the correctness of the method under certain conditions, but:

How can we find the relevant regions?

$\hookrightarrow$ Try all possible regions $\rightsquigarrow$ irrelevant contributions are scaleless.

Automated by Mathematica code `asy.m`: Pak, A. Smirnov, Eur. Phys. J. C 71 (2011) 1626

```
AlphaRepExpand[{loop momenta}, {list of denominators},
  {replacements for kinematic invariants}, {scaling of parameters}]
```

- Expansion at level of Feynman-parameter integrals.
- Uses geometric interpretation of integral (details $\rightsquigarrow$ paper).
- Detects non-scaleless contributions.
- Works well, but fails to detect potential and Glauber regions.

Why does `asy.m` fail to detect potential regions?

Example: threshold expansion, $y = m^2 - \frac{q^2}{4} \to 0$:

$$F = \int \frac{\mathrm{D}k}{(k^2 - m^2)\left((k-q)^2 - m^2\right)} = \mu^{2\epsilon} e^{\epsilon\gamma_E} \Gamma(\epsilon) \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2\, \delta(1 - \sum x_i)\, (x_1 + x_2)^{2\epsilon - 2}}{\left[m^2 (x_1 - x_2)^2 + 4y\, x_1 x_2\right]^\epsilon}$$

$\hookrightarrow$ Feynman-parameter representation (where argument of δ-function may vary)

Relevant regions (specified by scaling relations for parameters x_1, x_2):

- **hard** (h): $x_1 \sim y^0$, $x_2 \sim y^0$
- **potential** (p): $x_1 + x_2 \sim y^0$, $x_1 - x_2 \sim y^{1/2}$ $\rightsquigarrow$ not found by `asy.m`!

$\hookrightarrow$ Only regions with simple scalings $x_i \sim y^{v_i}$ found!

New version: `asy2.m`

B.J., A. Smirnov, V. Smirnov, arXiv:1206.0546
`http://www-ttp.particle.uni-karlsruhe.de/~asmirnov/Tools-Regions.htm`

automatically eliminates cancellations between parameters by

- splitting the integral at the critical points,
- performing variable transformations:

$$\int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2\, \delta(1 - \sum x_i)\, (x_1 + x_2)^{2\epsilon - 2}}{\left[m^2 (x_1 - x_2)^2 + 4y\, x_1 x_2\right]^\epsilon} = \int_0^\infty \frac{\mathrm{d}x_1' \mathrm{d}x_2'\, \delta(1 - \sum x_i')\, (x_1' + x_2')^{2\epsilon - 2}}{\left[m^2 {x_2'}^2 + y\, x_1' (x_1' + 2x_2')\right]^\epsilon}$$

Regions after variable transformation:

$$F = \int \frac{\mathrm{D}k}{(k^2 - m^2)\big((k-q)^2 - m^2\big)} = \mu^{2\epsilon} e^{\epsilon\gamma_E} \Gamma(\epsilon) \int_0^\infty \frac{\mathrm{d}x_1' \mathrm{d}x_2' \, \delta(1 - \sum x_i') \, (x_1' + x_2')^{2\epsilon-2}}{\big[m^2 {x_2'}^2 + y \, x_1' (x_1' + 2x_2')\big]^\epsilon}$$

- **hard** (h): $x_1' \sim y^0$, $x_2' \sim y^0$
- **potential** (p): $x_1' \sim y^0$, $x_2' \sim y^{1/2}$

$\hookrightarrow$ no cancellations $\rightsquigarrow$ simple scalings $x_i' \sim y^{v_i}$ $\Rightarrow$ found by `asy.m` / `asy2.m` ✓

Usage of new features in `asy2.m`: option `PreResolve`

```
AlphaRepExpand[{k}, {k^2 - m^2, (k-q)^2 - m^2},
  {q^2 -> 4*(m^2 - y)}, {m -> 1, y -> x}, PreResolve -> True]
```

- automatically detects all regions
- prints the corresponding variable transformations $x_{1,2} \to x_{1,2}'$

Glauber regions:

- cancellations like $(x_1 - x_2)(x_3 - x_4)$
- automatically treated by `asy2.m`

V Summary

Expansion by regions: foundation and generalization

B.J., JHEP 12 (2011) 076

- Conditions for regions (+ corresponding expansions & domains) established.
- Identity proven $\rightsquigarrow$ relates exact integral to sum of expanded terms:

$$F = \left\{ \begin{matrix} \text{single} \\ \text{expansions} \end{matrix} \right\} - \left\{ \begin{matrix} \text{double} \\ \text{expansions} \end{matrix} \right\} + \left\{ \begin{matrix} \text{triple} \\ \text{expansions} \end{matrix} \right\} - \ldots$$

$\hookrightarrow$ valid independent of the choice of regularization

- Identity includes overlap contributions with multiple expansions

  $\hookrightarrow$ can be scaleless $\rightsquigarrow$ known recipe for expansion by regions ✓

  or relevant (depending on regularization) $\rightsquigarrow$ generalization of known recipe.

Automated search for regions with `asy2.m`

B.J., A. Smirnov, V. Smirnov, arXiv:1206.0546

$\hookrightarrow$ automatic detection of the relevant regions for a given integral.

- Original algorithm of `asy.m` extended by automatic variable transformation.
- `asy2.m` reveals all relevant regions of a (multi-)loop integral – or issues a warning.

  $\hookrightarrow$ Also finds potential & Glauber regions now.

- `http://www-ttp.particle.uni-karlsruhe.de/~asmirnov/Tools-Regions.htm`

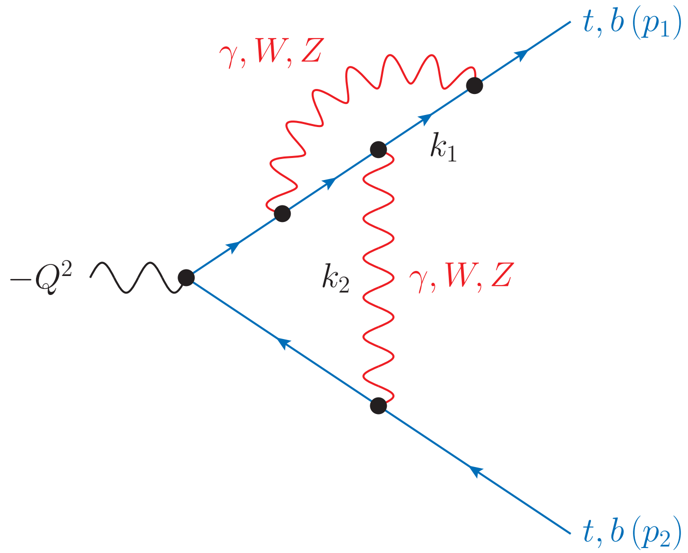
Extra slides

The expansion by regions has been applied successfully to many complicated loop integrals.

"Real-life" example

2-loop vertex integral in the high-energy limit

Denner, B.J., Pozzorini '08



$$Q^2 \gg m_t^2 \sim M_{W,Z}^2$$

$\hookrightarrow$ 9 relevant regions: [labelled "$(k_1 - k_2)$"]

$(h-h)$, $(1c-h)$, $(h-2c)$,
$(1c-1c)$, $(1c-2c)$, $(2c-2c)$,
$(1c-2uc)$, $(2uc-2uc)$, $(us-2c)$

- next-to-leading-logarithmic result obtained:
  $\alpha^2 \left\{L^3,\, L^2/\epsilon,\, L/\epsilon^2,\, 1/\epsilon^3\right\}$, where $L = \ln(Q^2/M_W^2)$
- cross-checked with independent calculation based on sector decomposition

Practical note: how to find the relevant regions

- Look where the propagators have poles:
  - ⋆ Large-momentum example: $(k+p)^2 = 0$ at $k \sim p$, $\quad k^2 - m^2 = 0$ at $k \sim m$.
  - ⋆ Close the integration contour of one component (e.g. k^0, $k^\pm$).
    For all residues investigate the scaling of the components.

- Use Mellin–Barnes (MB) representations:

$$\frac{1}{(A+B)^n} = \frac{1}{\Gamma(n)} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2i\pi} \, \Gamma(n+z) \, \Gamma(-z) \, \frac{B^z}{A^{n+z}}$$

1. Evaluate the full (scalar) integral for generic propagator powers n_i in terms of multiple MB integrals.
2. Close MB contours involving the expansion parameter and extract the leading contributions.
3. The individual terms can be identified with corresponding regions by their homogeneous scaling with the expansion parameter depending on d and n_i.

[A subsequent expansion by regions often yields simpler expressions for the contributions.]

Practical note: how to find the relevant regions (2)

- Try all possible regions that you can imagine . . .
  If a region does not contribute, its integrals are scaleless.

- Automated by Mathematica code `asy.m`, Pak, A. Smirnov, Eur. Phys. J. C 71 (2011) 1626
  finds non-scaleless contributions automatically via geometric approach:

  ```
  AlphaRepExpand[{k}, {(k+p)^2, k^2-m^2}, {p^2->1}, {m^2->x}]
  ```

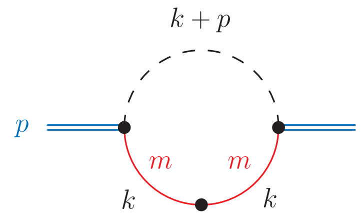
  Expansion based on Feynman-parameter integral $\rightsquigarrow$ result: list of regions with scalings of Feynman parameters in powers of the expansion parameter

  First version of `asy.m`: potential & Glauber regions not found
  $\hookrightarrow$ solved by update `asy2.m` B.J., A. Smirnov, V. Smirnov, arXiv:1206.0546

- When a region is missing, the total result is often (but not always) more singular than it should be. $\rightsquigarrow$ Important cross-check, but no guarantee!

Large-momentum expansion:

$$F = \int \frac{\mathrm{D}k}{(k+p)^2\,(k^2-m^2)^2}$$



Expansion to all orders in $\frac{m^2}{p^2}$

- **hard:** $\sum_i T_i^{(h)} \frac{1}{(k^2-m^2)^2} = \sum_{i=0}^{\infty}(1+i)\frac{(m^2)^i}{(k^2)^{2+i}}$ $\quad [(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)]$

$$\hookrightarrow\ F^{(h)} = \frac{1}{p^2}\left(\frac{\mu^2}{-p^2}\right)^{\epsilon} \frac{e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)\,\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i=0}^{\infty}\left(\frac{m^2}{p^2}\right)^i \frac{(2\epsilon)_i}{i!}$$

$$= \frac{1}{p^2}\left[-\frac{1}{\epsilon} + \ln\left(\frac{-p^2}{\mu^2}\right) + 2\ln\left(1-\frac{m^2}{p^2}\right)\right] + \mathcal{O}(\epsilon)$$

- **soft:** $\sum_j T_j^{(s)} \frac{1}{(k+p)^2} = \sum_{j_1,j_2=0}^{\infty} \frac{(j_1+j_2)!}{j_1!\,j_2!}\,\frac{(-2k\cdot p)^{j_1}\,(-k^2)^{j_2}}{(p^2)^{1+j_1+j_2}}$

$$\hookrightarrow\ F^{(s)} = \frac{1}{p^2}\left(\frac{\mu^2}{m^2}\right)^{\epsilon} e^{\epsilon\gamma_E}\,\Gamma(\epsilon) \sum_{j=0}^{\infty}\left(\frac{m^2}{p^2}\right)^j \frac{(\epsilon)_j}{(1-\epsilon)_j}$$

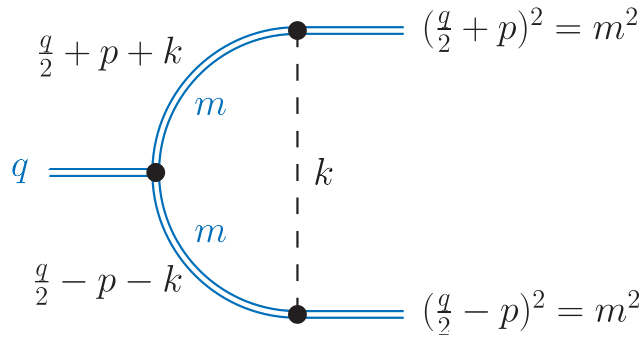
$$= \frac{1}{p^2}\left[\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right) - \ln\left(1-\frac{m^2}{p^2}\right)\right] + \mathcal{O}(\epsilon)$$

Full result F exactly reproduced:

$$F = F^{(h)} + F^{(s)} = \frac{1}{p^2}\left[\ln\left(\frac{-p^2}{m^2}\right) + \ln\left(1-\frac{m^2}{p^2}\right)\right] + \mathcal{O}(\epsilon) \quad \checkmark$$

Example with 3 regions: threshold expansion for heavy-particle pair production

Regions analyzed in Beneke, Smirnov, NPB 522 (1998) 321



Centre-of-mass system: $(q^\mu)=(q_0,\vec{0})$, $(p^\mu)=(0,\vec{p})$

Close to threshold: $q^2 \approx (2m)^2 \Rightarrow$ $q^2 \gg |p^2|$ or $q_0 \gg |\vec{p}|$

$$F=\int \frac{\mathrm{D}k}{(k^2+q_0k_0-2\vec{p}\cdot\vec{k})\,(k^2-q_0k_0-2\vec{p}\cdot\vec{k})\,k^2}$$

Relevant regions:

- **hard** (h): $k_0, |\vec{k}| \sim q_0 \Rightarrow$ expand $\sum_j T_j^{(h)}$ in $D_h=\left\{k\in\mathbb{R}^d : |k_0|\gg|\vec{p}| \text{ or } |\vec{k}|\gg|\vec{p}|\right\}$
- **soft** (s): $k_0, |\vec{k}| \sim |\vec{p}| \Rightarrow$ expand $\sum_j T_j^{(s)}$ in $D_s=\left\{k\in\mathbb{R}^d : |\vec{k}|\lesssim|k_0|\lesssim|\vec{p}|\right\}$
- **potential** (p): $k_0\sim\frac{\vec{p}^{\,2}}{q_0}$, $|\vec{k}|\sim|\vec{p}| \Rightarrow \sum_j T_j^{(p)}$ in $D_p=\left\{k\in\mathbb{R}^d : |k_0|\ll|\vec{k}|\lesssim|\vec{p}|\right\}$

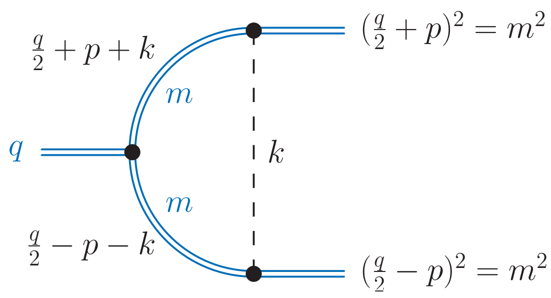
[no explicit boundaries needed]

↪ The expansion $T^{(x)}\equiv\sum_j T_j^{(x)}$ converges for $k\in D_x$ $(x=h,s,p)$.

↪ $D_h\cup D_s\cup D_p=\mathbb{R}^d$ $\quad[D_h\cap D_s=D_h\cap D_p=D_s\cap D_p=\emptyset]$

↪ The expansions $T^{(h)}, T^{(s)}, T^{(p)}$ commute with each other.

Threshold expansion (2)



Similar transformations as for the large-momentum example yield the following **identity:**

$$F = F^{(h)} + \underbrace{F^{(s)}}_{=0} + F^{(p)} - \Big(\underbrace{F^{(h,s)}}_{=0} + \underbrace{F^{(h,p)}}_{=0} + \underbrace{F^{(s,p)}}_{=0}\Big) + \underbrace{F^{(h,s,p)}}_{=0 \text{ (scaleless)}}$$

with

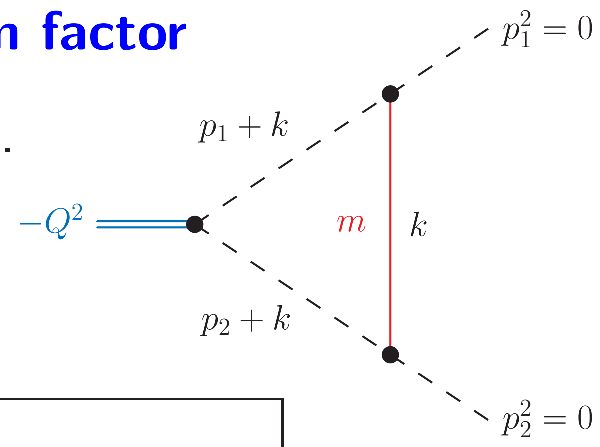
$$F^{(h)} = -\frac{2\, e^{\epsilon\gamma_E}\,\Gamma(\epsilon)}{q^2}\left(\frac{4\mu^2}{q^2}\right)^{\epsilon} \sum_{j=0}^{\infty} \left(-\frac{4p^2}{q^2}\right)^{j} \frac{(1+\epsilon)_j}{j!\,(1+2\epsilon+2j)}$$

$$F^{(p)} = \frac{e^{\epsilon\gamma_E}\,\Gamma(\frac{1}{2}+\epsilon)\,\sqrt{\pi}}{2\epsilon\,\sqrt{q^2\,(p^2-i0)}}\left(\frac{\mu^2}{p^2-i0}\right)^{\epsilon} \qquad \text{[higher orders vanish]}$$

Exact result reproduced:

$$F^{(h)} + F^{(p)} = F = \frac{e^{\epsilon\gamma_E}\,\Gamma(\epsilon)}{2p^2}\left(\frac{\mu^2}{p^2-i0}\right)^{\epsilon} {}_2F_1\!\left(\frac{1}{2}, 1+\epsilon; \frac{3}{2}; -\frac{q^2}{4p^2} - i0\right) \quad \checkmark$$

Example with non-commuting expansions: Sudakov form factor



Cannot always choose expansions which commute with each other.

Sudakov limit: $-(p_1 - p_2)^2 = \boxed{Q^2 \gg m^2}$

$$F = \int \frac{\mathrm{D}k}{(k^+k^- - \vec{k}_\perp^2 + Qk^+)^{1+\delta}\,(k^+k^- - \vec{k}_\perp^2 + Qk^-)^{1-\delta}\,(k^+k^- - \vec{k}_\perp^2 - m^2)}$$

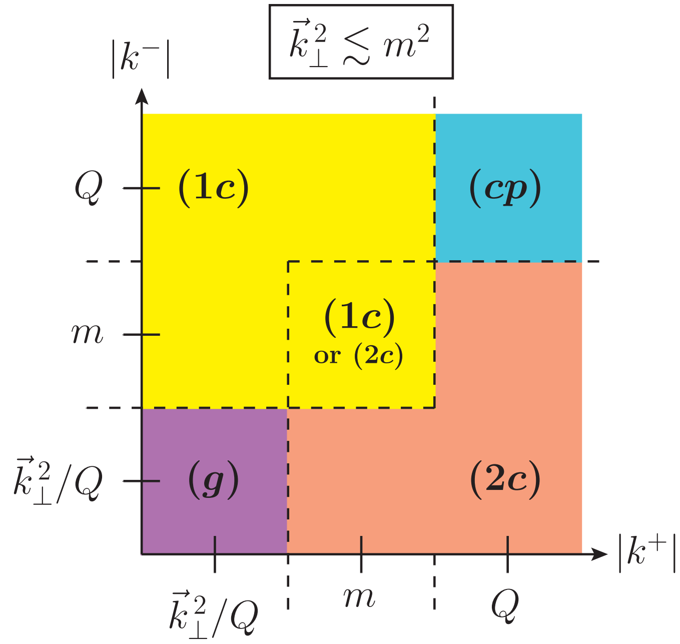
$\hookrightarrow$ analytic regulator $\delta \to 0$ [light-cone coordinates: $2p_{1,2} \cdot k = Qk^\pm$, $p_{1,2} \cdot k_\perp = 0$]

Regions & domains:

- **hard** (h): $k^+, k^-, |\vec{k}_\perp| \sim Q \Rightarrow D_h = \left\{k \in \mathbb{R}^d : \vec{k}_\perp^2 \gg m^2\right\}$
- **1-collinear** $(1c)$: $k^+ \sim \frac{m^2}{Q}$, $k^- \sim Q$, $|\vec{k}_\perp| \sim m$
- **2-collinear** $(2c)$: $k^+ \sim Q$, $k^- \sim \frac{m^2}{Q}$, $|\vec{k}_\perp| \sim m$
- **Glauber** (g): $k^+, k^- \sim \frac{m^2}{Q}$, $|\vec{k}_\perp| \sim m$
- **collinear-plane** (cp): $k^+, k^- \sim Q$, $|\vec{k}_\perp| \sim m$

  $\hookrightarrow$ "artificial" region to ensure $\cup_x D_x = \mathbb{R}^d$

[No soft region needed: $T^{(s)} \equiv T^{(1c)} T^{(2c)}$]



Most expansions commute, but $T^{(g)} T^{(cp)} \neq T^{(cp)} T^{(g)}$!

Sudakov form factor (2)

$T^{(g)}T^{(cp)} \neq T^{(cp)}T^{(g)}$ $\rightsquigarrow$ Construct **identity** avoiding combination of (g) and (cp):

$$
\begin{aligned}
F &= F^{(h)} + F^{(1c)} + F^{(2c)} + F^{(g)} + F^{(cp)} \\
&\quad - \Big(F^{(h,1c)} + F^{(h,2c)} + F^{(h,g)} + F^{(h,cp)} + F^{(1c,2c)} + F^{(1c,g)} + F^{(1c,cp)} + F^{(2c,g)} + F^{(2c,cp)}\Big) \\
&\quad + F^{(h,1c,2c)} + F^{(h,1c,g)} + F^{(h,1c,cp)} + F^{(h,2c,g)} + F^{(h,2c,cp)} + F^{(1c,2c,g)} + F^{(1c,2c,cp)} \\
&\quad - \Big(F^{(h,1c,2c,g)} + F^{(h,1c,2c,cp)}\Big) + F^{\text{extra}}_{cp\leftarrow g} + F^{\text{extra}}_{g\leftarrow cp}
\end{aligned}
$$

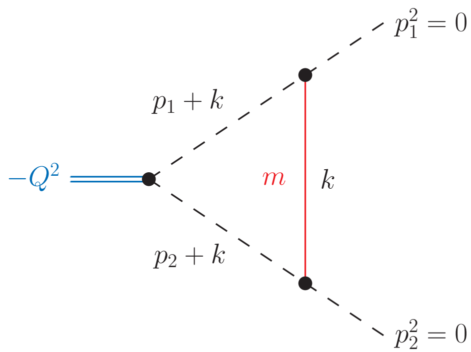
Usual terms:

- no combination of (g) and (cp)
- $F^{(g)}$, $F^{(cp)}$ and all overlap contributions are scaleless (with analytic regularization)

Extra terms:

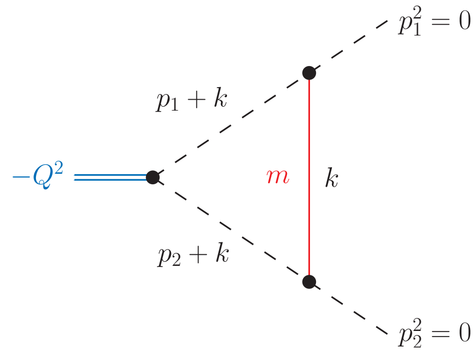
- $F^{\text{extra}}_{cp\leftarrow g}$ involves $T^{(cp)}T^{(g)}$ integrated over $k \in D_{cp}$,
- $F^{\text{extra}}_{g\leftarrow cp}$ involves $T^{(g)}T^{(cp)}$ integrated over $k \in D_g$,

plus all combinations of $T^{(h)}, T^{(1c)}, T^{(2c)}$, with alternating signs.



Both extra terms cancel at the integrand level,
because $T^{(1c)}T^{(g)}T^{(cp)} = T^{(g)}T^{(cp)}$ and similar relations hold.

Sudakov form factor (3)



Both extra terms cancel at the integrand level:

$$F^{\text{extra}}_{g\leftarrow cp} = \int_{k\in D_g} \mathrm{D}k \Big(-1 + T^{(h)} + T^{(1c)} + T^{(2c)} - T^{(h,1c)} - T^{(h,2c)} - T^{(1c,2c)} + T^{(h,1c,2c)}\Big) T^{(g)} T^{(cp)} I$$

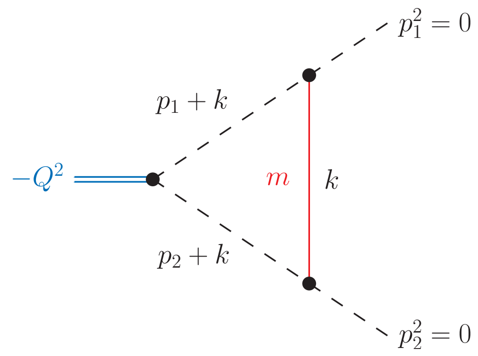
$$= (-1+3-3+1) \int_{k\in D_g} \mathrm{D}k\, T^{(g)} T^{(cp)} I = 0$$

because $T^{(x)} T^{(g)} T^{(cp)} = T^{(g)} T^{(cp)}$ $\forall x \in \{h, 1c, 2c\}$.

Similarly: $F^{\text{extra}}_{cp\leftarrow g} = 0$ because $T^{(x)} T^{(cp)} T^{(g)} = T^{(cp)} T^{(g)}$ $\forall x \in \{1c, 2c\}$.

[The extra terms must cancel $\rightsquigarrow$ otherwise dependence on boundaries of D_g, D_{cp}.]

Sudakov form factor (4)



Omitting scaleless contributions and vanishing extra terms:

$$\boxed{F = F^{(h)} + F^{(1c)} + F^{(2c)}}$$

Regions explicitly evaluated to all orders in $\frac{m^2}{Q^2}$: [omitting $\mathcal{O}(\delta)$ and $\mathcal{O}(\epsilon)$]

$$F^{(h)} = -\frac{1}{Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\left\{\frac{1}{\epsilon^2} - \frac{2}{\epsilon}\ln\left(1-\frac{m^2}{Q^2}\right) + \ln^2\left(1-\frac{m^2}{Q^2}\right) - 2\,\mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) - \frac{\pi^2}{12}\right\}$$

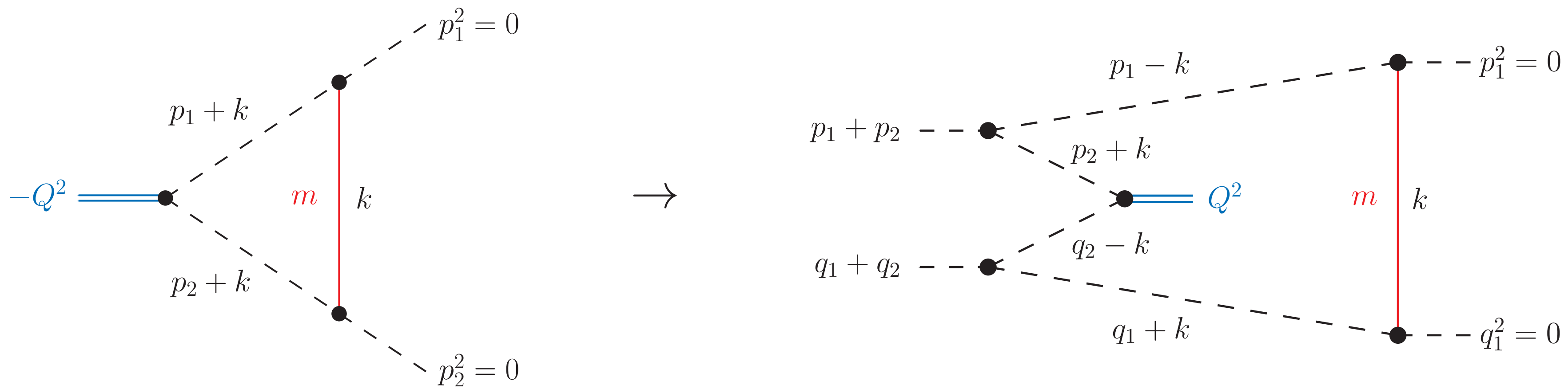
$$F^{(1c)}, F^{(2c)} = -\frac{1}{2Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\left\{\pm\frac{1}{\delta}\left[\frac{1}{\epsilon} + \ln\frac{Q^2}{m^2} - \ln\left(1-\frac{m^2}{Q^2}\right)\right] - \frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\left(1-\frac{m^2}{Q^2}\right)\right.$$
$$\left. + \frac{1}{2}\ln^2\frac{Q^2}{m^2} + \ln\frac{Q^2}{m^2}\ln\left(1-\frac{m^2}{Q^2}\right) - \ln^2\left(1-\frac{m^2}{Q^2}\right) + \mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) + \frac{5}{12}\pi^2\right\}$$

$\hookrightarrow$ $F^{(1c)}$ and $F^{(2c)}$ are not separately finite for $\delta \to 0$, but their sum is.

Agreement with exact result:

$$F = -\frac{1}{Q^2}\left\{\frac{1}{2}\ln^2\frac{Q^2}{m^2} + \ln\frac{Q^2}{m^2}\ln\left(1-\frac{m^2}{Q^2}\right) - \mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) + \frac{\pi^2}{3}\right\} \quad \checkmark$$

Sudakov form factor → 5-point integral with Glauber contribution



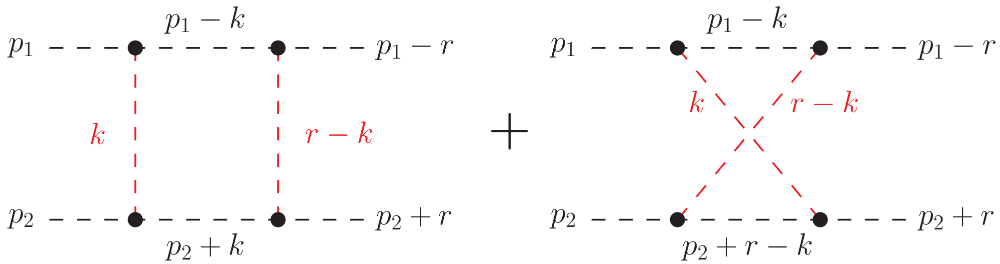
- collinear propagators "doubled", but expansions equivalent
- same regions & domains
- "double" propagators ⇝ Glauber contribution present (even with analytic regularization)
- leading contributions:

$$F_0^{(g)} \propto \frac{1}{(Q^2)^3} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left(\frac{m^2}{Q^2}\right)^{-2-\epsilon}$$

$$F_0^{(1c)}, F_0^{(2c)} \propto \frac{1}{(Q^2)^3} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left(\frac{m^2}{Q^2}\right)^{-1-\epsilon}$$

$$F_0^{(h)} \propto \frac{1}{(Q^2)^3} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon}$$

Example with relevant overlap contributions: forward scattering with small momentum exchange



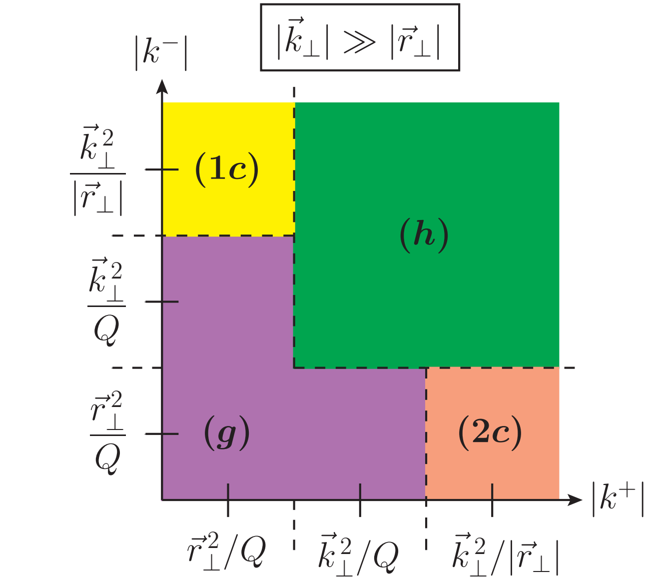
Two light-like particles with large centre-of-mass energy exchange a small momentum r:

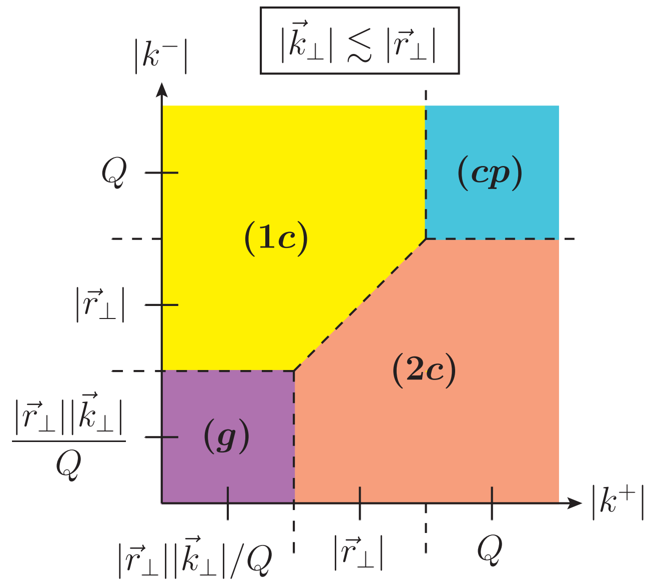
$$p_1^2 = (p_1 - r)^2 = p_2^2 = (p_2 + r)^2 = 0$$

$$(p_1 + p_2)^2 = \boxed{Q^2 \gg \vec{r}_\perp^{\,2}}, \quad r^\pm \approx \mp \frac{\vec{r}_\perp^{\,2}}{Q}$$

Symmetrize integral under $k \leftrightarrow r - k$
$\hookrightarrow$ avoids divergences at $|k^\pm| \to \infty$ under expansion.

$$F = \frac{1}{2} \int \frac{\mathrm{D}k}{k^2 (r-k)^2} \left(\frac{1}{\left((p_1 - k)^2\right)^{1+\delta}} + \frac{1}{\left((p_1 - r + k)^2\right)^{1+\delta}} \right) \times \left(\frac{1}{\left((p_2 + k)^2\right)^{1-\delta}} + \frac{1}{\left((p_2 + r - k)^2\right)^{1-\delta}} \right)$$

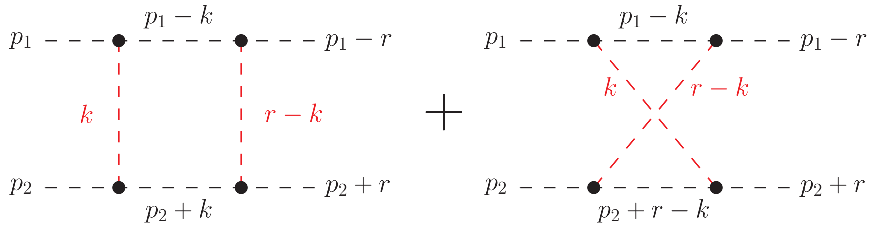




Regions: same as for Sudakov form factor (scaling with $m \to |\vec{r}_\perp|$),

Domains: similar (but more involved for $|\vec{k}_\perp| \gg |\vec{r}_\perp|$)

Forward scattering (2)



Same identity as for Sudakov form factor:

$$F = F^{(h)} + F^{(1c)} + F^{(2c)} + F^{(g)} + F^{(cp)}$$
$$- \Big(F^{(h,1c)} + F^{(h,2c)} + F^{(h,g)} + F^{(h,cp)} + F^{(1c,2c)} + F^{(1c,g)} + F^{(1c,cp)} + F^{(2c,g)} + F^{(2c,cp)} \Big)$$
$$+ F^{(h,1c,2c)} + F^{(h,1c,g)} + F^{(h,1c,cp)} + F^{(h,2c,g)} + F^{(h,2c,cp)} + F^{(1c,2c,g)} + F^{(1c,2c,cp)}$$
$$- \Big(F^{(h,1c,2c,g)} + F^{(h,1c,2c,cp)} \Big)$$

With analytic regulator $\delta \to 0$: $\boxed{F_0 = F_0^{(1c)} + F_0^{(2c)}}$ [$F_0^{(h)}$ suppressed, others scaleless]

$$F_0^{(1c)} = F_0^{(2c)} = \frac{1}{2} \frac{i\pi}{Q^2 \vec{r}_\perp^{\,2}} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}} \right)^\epsilon \frac{e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}$$

Without analytic regularization ($\delta = 0$): [all terms are still well-defined]

$$\boxed{F_0 = F_0^{(1c)} + F_0^{(2c)} + F_0^{(g)} - \Big(F_0^{(1c,2c)} + F_0^{(1c,g)} + F_0^{(2c,g)} \Big) + F_0^{(1c,2c,g)}}$$

$$F_0^{(x,\dots)} = \frac{i\pi}{Q^2 \vec{r}_\perp^{\,2}} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}} \right)^\epsilon \frac{e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} \qquad \forall\,\{x,\dots\} \subset \{1c, 2c, g\}$$

↪ consistent results independent of regularization: $\frac{1}{2} + \frac{1}{2} = 1 + 1 + 1 - (1 + 1 + 1) + 1$ ✓

↪ agreement with leading-order expansion of full result

The general formalism (details)

Identities as in the examples are generally valid, under some conditions.

Consider

- a (multiple) integral $F = \int \mathrm{D}k\, I$ over the domain D (e.g. $D = \mathbb{R}^d$),
- a set of N regions $R = \{x_1, \ldots, x_N\}$,
- for each region $x \in R$ an expansion $T^{(x)} = \sum_j T_j^{(x)}$ which converges absolutely in the domain $D_x \subset D$.

Conditions

- $\bigcup_{x \in R} D_x = D \qquad [D_x \cap D_{x'} = \emptyset\ \forall x \neq x']\,.$
- Some of the expansions commute with each other.
  Let $R_\mathsf{c} = \{x_1, \ldots, x_{N_\mathsf{c}}\}$ and $R_\mathsf{nc} = \{x_{N_\mathsf{c}+1}, \ldots, x_N\}$ with $1 \leq N_\mathsf{c} \leq N$.
  Then: $T^{(x)} T^{(x')} = T^{(x')} T^{(x)} \equiv T^{(x,x')}\ \forall x \in R_\mathsf{c}\,,\ x' \in R\,.$
- Every pair of non-commuting expansions is invariant under some expansion from R_c:
  $\forall x_1', x_2' \in R_\mathsf{nc},\ x_1' \neq x_2',\ \exists\, x \in R_\mathsf{c} : T^{(x)} T^{(x_2')} T^{(x_1')} = T^{(x_2')} T^{(x_1')}\,.$
- $\exists$ regularization for singularities, e.g. dimensional (+ analytic) regularization.
  $\hookrightarrow$ All expanded integrals and series expansions in the formalism are well-defined.

The general formalism (2)

Under these conditions, the following **identity** holds: $\left[F^{(x,\dots)} \equiv \sum_{j,\dots} \int \mathrm{D}k\, T_{j,\dots}^{(x,\dots)} I\right]$

$$F = \sum_{x\,\in\,R}^{\langle R_\mathsf{c}+1\rangle} F^{(x)} - \sum_{\{x'_1,\,x'_2\}\,\subset\,R}^{\langle R_\mathsf{c}+1\rangle} F^{(x'_1,x'_2)} + \ldots - (-1)^n \sum_{\{x'_1,\,\ldots,\,x'_n\}\,\subset\,R} F^{(x'_1,\ldots,x'_n)} + \ldots + (-1)^{N_\mathsf{c}} \sum_{x'\,\in\,R_\mathsf{nc}} F^{(x',x_1,\ldots,x_{N_\mathsf{c}})}$$

where the sums run over subsets $\{x'_1,\ldots\}$ containing at most one region from R_nc.

Comments

- This identity is exact when the expansions are summed to all orders. ✓
  Leading-order approximation for F $\rightsquigarrow$ dropping higher-order terms.
- It is independent of the regularization (dim. reg., analytic reg., cut-off, infinitesimal masses/off-shellness, ...) as long as all individual terms are well-defined.
- Usually regions & regularization are chosen such that multiple expansions $F^{(x'_1,\ldots,x'_n)}$ $(n \geq 2)$ are scaleless and vanish.
  [✓ if each $F_0^{(x)}$ is a *homogeneous* function of the expansion parameter with *unique scaling*.]
- If $\exists\, F^{(x'_1,x'_2,\ldots)} \neq 0$ $\rightsquigarrow$ relevant overlap contributions ($\rightarrow$ "zero-bin subtractions").
  They appear e.g. when avoiding analytic regularization in SCET. e.g. Manohar, Stewart '06; Chiu, Fuhrer, Hoang, Kelley, Manohar '09; ...

Automated search for regions with `asy2.m` (details)

Practical question: How to find the relevant regions?

- Look where the integrand has poles or singularities.
- Extract (form of) expansion terms using Mellin–Barnes representations.
- Try all possible regions $\rightsquigarrow$ irrelevant contributions are scaleless.
  - $\hookrightarrow$ avoid double-counting of regions with equivalent expansions
  - $\hookrightarrow$ automatic identification of regions easier in parametric integrals

Example: threshold expansion, $y = m^2 - \frac{q^2}{4} \to 0$:

$$F = \int \frac{\mathrm{D}k}{(k^2 - m^2)\big((k-q)^2 - m^2\big)} = \mu^{2\epsilon} e^{\epsilon\gamma_E} \Gamma(\epsilon) \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2\, \delta(1 - \sum x_i)\, (x_1 + x_2)^{2\epsilon-2}}{\big[m^2 (x_1 - x_2)^2 + 4y\, x_1 x_2\big]^\epsilon}$$

$\hookrightarrow$ Feynman-parameter representation (where argument of δ-function may vary)

Regions specified by scaling relations for parameters x_1, x_2:

- **hard** (h): $x_1 \sim y^0$, $x_2 \sim y^0$
- **potential** (p): $x_1 + x_2 \sim y^0$, $x_1 - x_2 \sim y^{1/2}$

Geometric approach for expansion by regions

`Mathematica` code `asy.m`: Pak, A. Smirnov, Eur. Phys. J. C 71 (2011) 1626

- Each monomial from $(x_1 + x_2) \cdot \left[m^2(x_1 - x_2)^2 + 4y\, x_1 x_2\right]$
  $\hookrightarrow$ point in 3-dimensional vector space describing its scaling in powers of y, x_1, x_2.
- Calculate convex hull of these points using `Qhull`. `http://www.qhull.org`
- Facets of convex hull determine scalings $x_i \sim y^{v_i}$ of all regions with non-vanishing ($=$ non-scaleless) contributions.

$\hookrightarrow$ Hard region ($x_1, x_2 \sim y^0$) found, but potential region ($x_1 - x_2 \sim y^{1/2}$) not found!

New version: `asy2.m`

B.J., A. Smirnov, V. Smirnov, arXiv:1206.0546

`http://www-ttp.particle.uni-karlsruhe.de/~asmirnov/Tools-Regions.htm`

performs automatic change of variables to eliminate differences like $(x_1 - x_2)$:

- for $x_1 \le x_2$: $x_1 = x_1'/2$, $x_2 = x_2' + x_1'/2$
- for $x_1 \ge x_2$: $x_2 = x_1'/2$, $x_1 = x_2' + x_1'/2$

$$\int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2\, \delta(1 - \sum x_i)\, (x_1 + x_2)^{2\epsilon-2}}{\left[m^2(x_1 - x_2)^2 + 4y\, x_1 x_2\right]^\epsilon} = \int_0^\infty \frac{\mathrm{d}x_1' \mathrm{d}x_2'\, \delta(1 - \sum x_i')\, (x_1' + x_2')^{2\epsilon-2}}{\left[m^2 {x_2'}^2 + y\, x_1'(x_1' + 2x_2')\right]^\epsilon}$$

Usage of `asy2.m`

For (multi-)loop integrals:

$$F = \int \frac{\mathrm{D}k}{(k^2 - m^2)\big((k-q)^2 - m^2\big)} = \mu^{2\epsilon} e^{\epsilon\gamma_E}\Gamma(\epsilon) \int\limits_0^\infty \frac{\mathrm{d}x_1'\mathrm{d}x_2'\,\delta(1-\sum x_i')\,(x_1'+x_2')^{2\epsilon-2}}{\big[m^2 {x_2'}^2 + y\,x_1'(x_1'+2x_2')\big]^\epsilon}$$

```
AlphaRepExpand[{k}, {k^2 - m^2, (k-q)^2 - m^2},
  {q^2 -> 4*(m^2 - y)}, {m -> 1, y -> x}, PreResolve -> True]
```

automatically detects all regions

- **hard** (h): $x_1' \sim y^0$, $x_2' \sim y^0$ $\rightsquigarrow$ $T_0^{(h)} I = (x_1' + x_2')^{2\epsilon-2}\,(m^2 {x_2'}^2)^{-\epsilon}$
- **potential** (p): $x_1' \sim y^0$, $x_2' \sim y^{1/2}$ $\rightsquigarrow$ $T_0^{(p)} I = {x_1'}^{2\epsilon-2}\,(m^2 {x_2'}^2 + y\,{x_1'}^2)^{-\epsilon}$

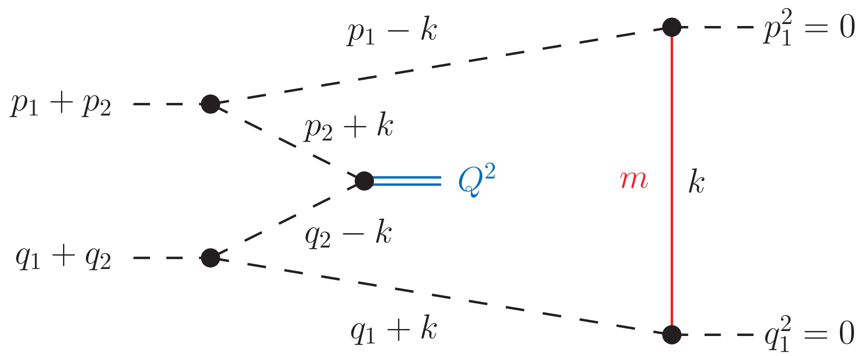
and prints the corresponding variable transformations $x_{1,2} \to x_{1,2}'$.

Also for general parametric integrals:

```
WilsonExpand[m^2*x2^2 + y*x1*(x1+2*x2), x1+x2,
  {x1, x2}, {m -> 1, y -> x}, Delta -> True]
```

Details of syntax & output descriptions $\rightsquigarrow$ paper B.J., A. Smirnov, V. Smirnov '12

Glauber regions with `asy2.m`



5-point integral with simplified kinematics:

$p_1 = p_2 = p$, $q_1 = q_2 = q$, $p^2 = q^2 = 0$,

$(p+q)^2 = Q^2 \gg m^2$

$$F = -\mu^{2\epsilon} e^{\epsilon\gamma_E} \Gamma(3+\epsilon) \int_0^\infty \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_5\, \delta(1-\sum x_i)\, (x_1 + \ldots + x_5)^{1+2\epsilon}}{\left[Q^2(x_2 - x_3)(x_4 - x_5) + m^2 x_1 (x_1 + \ldots + x_5) - i0\right]^{3+\epsilon}}$$

Glauber region present: $x_2 - x_3 \sim m^2$ or $x_4 - x_5 \sim m^2$

$\hookrightarrow$ 2-fold variable transformation to eliminate both differences $(x_2 - x_3)(x_4 - x_5)$

$\hookrightarrow$ performed automatically by `asy2.m`:

```
AlphaRepExpand[{k}, {k^2 - m^2, (p-k)^2, (p+k)^2, (q-k)^2, (q+k)^2},
  {p^2 -> 0, q^2 -> 0, p*q -> Q^2/2}, {Q -> 1, m^2 -> x},
  PreResolve -> True]
```

$\hookrightarrow$ finds all relevant regions (including variable transformations) ✓

Details about correspondence between regions in x_i and regions in k $\rightsquigarrow$ paper

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